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To Claudia

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Preface

This monograph gives a unified exposition of the *thermodynamic formalism* and some of its main extensions, with emphasis on their relation to *dimension theory* and *multifractal analysis* of dynamical systems. Not only are these natural playgrounds for nontrivial applications of the thermodynamic formalism, but are also major sources of inspiration for further developments of the theory.

In particular, we present the main results and main techniques in the interplay between the thermodynamic formalism, symbolic dynamics, dimension theory, and multifractal analysis. We also discuss selected topics of current research interest that until now were scattered in the literature (incidentally, more than two thirds of the material appears here for the first time in book form). This includes the discussion of some of the most significant recent results in the area as well as some of its open problems, in particular concerning dimension estimates for repellers and hyperbolic sets, dimension estimates or even formulas for the dimension of limit sets of geometric constructions, and the multifractal analysis of entropy and dimension spectra, in particular associated to nonconformal repellers. Undoubtedly, this selection, although quite conscious, also reflects a personal taste.

The dimension theory and the multifractal analysis of dynamical systems have progressively developed into an independent field of research during the last three decades. Nevertheless, despite a large number of interesting and nontrivial developments, only the case of *conformal* dynamics is completely understood. In the case of repellers this corresponds to assuming that the derivative of the map is a multiple of an isometry at every point. This property allowed Bowen in 1979 (in the particular case of quasi-circles) and then Ruelle in 1982 (in full generality) to develop a fairly complete theory for the dimension of repellers of conformal maps. Their work is strongly based on the thermodynamic formalism, earlier developed by Ruelle in 1973 for expansive transformations, and then by Walters in 1976 in full generality.

On the other hand, the study of the dimension of invariant sets of *nonconformal* maps unveiled several new phenomena, but it still lacks today a satisfactory general approach. In particular, we are often only able to establish dimension estimates instead of giving formulas for the dimension of the invariant sets. Thus, somemes the emphasis is on how to obtain sharp dimension estimates, starting essentially with the seminal work of Douady and Oesterlé in 1980, who devised an approach to cover an invariant set in a more optimal manner. Furthermore, it was early recognized, notably by Pesin and Pitskel' in 1984 (with the notion of topological pressure for noncompact sets) and by Falconer in 1988 (with his subadditive version of the thermodynamic formalism), that it would also be desirable to have an appropriate extension of the thermodynamic formalism in order to consider more general classes of invariant sets, and in particular invariant sets of nonconformal transformations. Most certainly, this is not foreign to the fact that virtually all known equations used to compute or estimate dimensions are appropriate versions of an equation introduced by Bowen in his study of quasi-circles that involves topological pressure, which is the most basic notion of the thermodynamic formalism.

The exposition is organized in four parts. The first part gives an introduction to the classical thermodynamic formalism and its relations to symbolic dynamics. Although everything is proven, we develop the theory in a pragmatic manner, only as much as needed for the following parts. The remaining three parts consider three different versions of the thermodynamic formalism, namely nonadditive, subadditive, and almost additive. In each of these parts we detail generously not only the most significant results in the area, some of them quite recent, but also some of the most substantial applications of the corresponding thermodynamic formalism to dimension theory and multifractal analysis of dynamical systems.

The nonadditive thermodynamic formalism, which is a considerable extension of the classical thermodynamic formalism, provides the most general setting and has a unifying role. The subadditive and the almost additive formalisms successively consider more special situations. As always in mathematics, when one makes further hypotheses, one can often establish additional results. Thus, it is not surprising that the nonadditive, subadditive, and almost additive thermodynamic formalisms are progressively richer. On the other hand, and this is a major motivation for such developments, the new hypotheses are still sufficiently general to allow a large number of nontrivial applications. This includes dimension estimates for nonconformal repellers, nonconformal hyperbolic sets, and limit sets of geometric constructions, as well as a multifractal analysis of entropy and dimension spectra of a large class of nonconformal repellers.

The book is directed to researchers as well as graduate students who wish to have a global view of the main results and main techniques in the area. It can also be used for graduate courses on the thermodynamic formalism and its extensions, with the optional discussion of some applications to dimension theory and multifractal analysis, or for graduate courses on special topics of dimension theory and multifractal analysis, with the discussion of the strictly necessary material from the thermodynamic formalism. We emphasize that with the exception of a few sections of survey type, the text is self-contained and all the results are included with detailed proofs. In particular, it can also be used for independent study.

There are no words that can adequately express my gratitude to Claudia Valls for her help, patience, encouragement, and inspiration during the preparation of this book. I acknowledge the support by FCT through the Center for Mathematical Analysis, Geometry, and Dynamical Systems of Instituto Superior Técnico.

Luis Barreira
Lisbon, May 2011

Chapter 1

Introduction

This book is dedicated to the thermodynamic formalism, its extensions, and its applications, with emphasis on the study of the relation to dimension theory and multifractal analysis of dynamical systems. We describe briefly in this chapter the historical origins and the principal elements of the research areas considered in the book. We also describe its contents. Finally, we recall in a pragmatic manner all the notions and results from dimension theory and ergodic theory that are needed later on.

1.1 Thermodynamic formalism and dimension theory

We describe in this section the historical origins of the thermodynamic formalism as well as of dimension theory and multifractal analysis of dynamical systems. In particular, we illustrate the rich interplay between these areas.

1.1.1 Classical thermodynamic formalism

The (mathematical) thermodynamic formalism has its roots in thermodynamics. For example, quoting from Gallavotti's foreword to Ruelle's book [166]:

> "Thermodynamics is still, as it always was, at the center of physics, the standard-bearer of successful science. As happens with many a theory, rich in applications, its foundations have been murky from the start and have provided a traditional challenge on which physicists and mathematicians alike have tested their latest skills."

Essentially, the thermodynamic formalism (following Ruelle's original expression) can be described as a rigorous study of certain mathematical structures inspired in thermodynamics. To differentiate it from the various extensions that are described in the book, we shall call it *classical thermodynamic formalism*.

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The notion of topological pressure, which is the most basic notion of the thermodynamic formalism, was introduced by Ruelle [164] for expansive transformations and by Walters [194] in the general case. For a continuous transformation $f\colon X \to X$ of a compact metric space, the *topological pressure* of a continuous function $\varphi\colon X \to \mathbb{R}$ (with respect to f) is defined by

$$P(\varphi) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \sup_E \sum_{x\in E} \exp \sum_{k=0}^{n-1} \varphi(f^k(x)),$$

where the supremum is taken over all (n,ε)-separated sets $E \subset X$ (see Section 2.1 for details). For example, taking $\varphi = 0$ we recover the notion of topological entropy.

The theory has progressively developed into a broad independent field, with many promising directions of research. In particular, the variational principle relating topological pressure to Kolmogorov–Sinai entropy was established by Ruelle [164] for expansive transformations and by Walters [194] in the general case. It says that

$$P(\varphi) = \sup_\mu \left(h_\mu(f) + \int_X \varphi \, d\mu \right), \tag{1.1}$$

where the supremum is taken over all f-invariant probability measures μ in X, and where $h_\mu(f)$ is the entropy with respect to the measure μ. The theory also includes a discussion of the existence and uniqueness of equilibrium and Gibbs measures, with the latter having a privileged relation to the Gibbs distributions of thermodynamics. For further developments of the thermodynamic formalism as well as a detailed discussion of its historical origins, we refer to the books [39, 108, 109, 149, 166, 195]. These developments also include directions of research that apparently are unrelated to the original motivation stemming from thermodynamics. We emphasize that it is entirely out of the scope of this book to provide any comprehensive exposition of the theory.

1.1.2 Dimension theory and multifractal analysis

We emphasize that in this book we are mainly concerned with the relation of the thermodynamic formalism and its extensions to the dimension theory of dynamical systems, which includes in particular the subfield of multifractal analysis. In other words, we do not consider topics of dimension theory that are not of a dynamical nature, of course independently of their importance. Roughly speaking, the main objective of the dimension theory of dynamical systems is to measure the complexity, from the dimensional point of view, of objects that remain invariant under the dynamics, such as invariant sets and measures. The first monograph that clearly took this point of view was Pesin's book [152], which describes the state-of-the-art up to 1997. We refer to the book [7] for a detailed description of many of the more recent results in the area.

The existence of a privileged relation between the thermodynamic formalism and the dimension theory of dynamical systems is due to the following fact. The unique solution s of the equation

$$P(s\varphi) = 0, \tag{1.2}$$

where φ is a certain function associated to a given invariant set, is often related to the Hausdorff dimension of the set. This equation was introduced by Bowen in [40] (in his study of quasi-circles) and is usually called Bowen's equation. It is also appropriate to call it the Bowen–Ruelle equation, taking not only into account the fundamental role of the thermodynamic formalism developed by Ruelle, but also his article [167] with a study of the Hausdorff dimension of the repellers of a conformal dynamics (this corresponds to assuming that the derivative of the map is a multiple of an isometry at every point). To a certain extent, the study of the dimension of hyperbolic sets is analogous. Indeed, assuming that the derivatives of the map along the stable and unstable directions are multiples of isometries, starting with the work of McCluskey and Manning in [133] it was possible to develop a sufficiently complete corresponding theory. However, there are nontrivial differences between the theory for repellers and the theory for hyperbolic sets. For example, each conformal repeller has a unique invariant measure of full dimension. On the other hand, unless some cohomology relations hold, there are no invariant measures of full dimension concentrated on a given conformal hyperbolic set.

Let us emphasize that virtually all known equations used to compute or to estimate the dimension of an invariant set, either of an invertible or a noninvertible dynamics, are particular cases of equation (1.2) or of an appropriate generalization. Nevertheless, despite these and many other significant developments, only the case of *conformal* dynamics is completely understood. In particular, many of the developments towards a nonconformal theory depend on each particular class of dynamics. On the other hand, this drawback of the theory is also a principal motivation for further developments and in particular for the extensions of the thermodynamic formalism that are presented in the book.

Now we turn to the theory of multifractal analysis. This is a subfield of the dimension theory of dynamical systems. Briefly, multifractal analysis studies the complexity of the level sets of any invariant local quantity obtained from a dynamical system. For example, one can consider Birkhoff averages, Lyapunov exponents, pointwise dimensions, or local entropies. These functions are usually only measurable and thus their level sets are rarely manifolds. Hence, in order to measure their complexity it is appropriate to use quantities such as the topological entropy or the Hausdorff dimension. The concept of multifractal analysis was suggested by Halsey, Jensen, Kadanoff, Procaccia and Shraiman in [84]. The first rigorous approach is due to Collet, Lebowitz and Porzio in [45] for a class of measures that are invariant under one-dimensional Markov maps. In [122], Lopes considered the measure of maximal entropy for hyperbolic Julia sets, and in [162], Rand studied Gibbs measures for a class of repellers. We refer the reader to the books [7, 152] for further references and for detailed expositions of parts of the theory. We also note that Morán [138] proposed a quite interesting approach to multifractal analysis in terms of multifractal decompositions obtained from multiplicative set functions.

1.1.3 Attractors in infinite-dimensional spaces

We discuss briefly in this section some motivations for the study of dimension in the context of the theory dynamical systems, mostly in connection with the theory of attractors in infinite-dimensional spaces.

The longtime behavior of many dynamical systems, such as those coming from delay differential equations and partial differential equations, can essentially be described in terms of a global attractor (see [3, 83, 189]). An important question, particularly in the context of infinite-dimensional systems, is how many degrees of freedom are necessary to specify the dynamics on the attractor. It turns out that a large class of attractors have finite Hausdorff dimension and even finite box dimension. Hence, the dynamics on the attractor is essentially finite-dimensional (see [3, 83, 189] for related discussions). In particular, Mañé [127] obtained the following result.

Theorem 1.1.1. *Let $f\colon E \to E$ be a C^1 map of a Banach space such that for each $x \in E$ the derivative $d_x f$ is the sum of a compact map and a contraction. Then every compact f-invariant set in E has finite upper box dimension.*

An analogous statement for the Hausdorff dimension was obtained earlier by Mallet-Paret [126] in the particular case of Hilbert spaces.

Moreover, particularly in the experimental study of attractors one often considers their projection into an Euclidean space. It is also possible to give conditions for the invertibility of the projection. In particular, the following result is also due to Mañé [127].

Theorem 1.1.2. *Let E be a Banach space and let $F \subset E$ be a p-dimensional subspace with $p < \infty$. For a residual set of the space of all continuous projections of E onto F (with respect to the topology induced by the operator norm), each projection is injective on a compact set $\Lambda \subset E$ provided that the product $\Lambda \times \Lambda$ has Hausdorff dimension less then $p - 1$.*

For an arbitrary projection of a compact subset of a Banach space, Hunt and Kaloshin [95] showed that typically (in the sense of prevalence in [96]) the projection is injective and has Hölder continuous inverse. Earlier results on the Hölder continuity of the inverse are due to Ben-Artzi, Eden, Foias and Nikolaenko [29] in $\mathbb{R}^n$ and to Foias and Olson [70] in Hilbert spaces. These results estimate how much the dimension of the set can decrease under the projection.

1.2 Extensions and applications

We describe in this section the main elements of the three extensions of the classical thermodynamic formalism that are discussed in the book, namely the nonadditive, the subadditive, and the almost additive thermodynamic formalisms. We also describe briefly some of the nontrivial applications of each extension, in particular to the dimension of repellers and hyperbolic sets, the dimension of limit sets of geometric constructions, and the multifractal analysis of entropy and dimension spectra.

1.2.1 Nonadditive formalism and dimension estimates

The nonadditive thermodynamic formalism was introduced by Barreira in [5]. It is a generalization of the classical thermodynamic formalism, in which the topological pressure $P(\varphi)$ of a continuous function φ (with respect to a given dynamics on a compact metric space) is replaced by the topological pressure $P(\Phi)$ of a sequence of continuous functions $\Phi = (\varphi_n)_{n\in\mathbb{N}}$. The nonadditive thermodynamic formalism contains as a particular case a new formulation of the subadditive thermodynamic formalism earlier introduced by Falconer in [56]. For additive sequences and arbitrary sets, it recovers the notion of topological pressure introduced by Pesin and Pitskel' in [153], and the notions of lower and upper capacity topological pressures introduced by Pesin in [151]. It also gives an equivalent description of the notion of topological pressure for compact sets introduced by Ruelle in [164] in the case of expansive maps, and by Walters in [194] in the general case.

Among the main motivations for the nonadditive thermodynamic formalism are certain applications to a much more general class of invariant sets in the context of the dimension theory of dynamical systems. Indeed, while the study of the dimension of invariant sets of *nonconformal* maps unveiled several new phenomena, it still lacks today a satisfactory general approach, both for repellers and for hyperbolic sets. In particular, most authors make additional assumptions that essentially avoid two main types of difficulties. The first difficulty is the lack of a clear separation between different Lyapunov directions, together with a possible small regularity of the associated distributions (or the associated holonomies). Typically, these distributions are only Hölder continuous, which causes that in general it is impossible to add the dimensions along various distributions. This strongly contrasts to what happens for hyperbolic sets of a conformal dynamics, in which case the stable and unstable holonomies are Lipschitz. The second difficulty is the existence of number-theoretical properties that may cause a variation of the Hausdorff dimension with respect to a certain typical value (such as that obtained by Falconer in [55]; see Theorem 5.3.5). Other authors have obtained results not for a specific invariant set, but instead for almost all invariant sets in a given parameterized family. Unfortunately, sometimes it is quite difficult to determine what happens for each specific value of the parameter, if at all possible.

These difficulties cause that in the general case of nonconformal maps, at the present stage of the theory we are often only able to establish dimension estimates instead of giving formulas for the dimension of an invariant set. Thus, sometimes the emphasis is on how to obtain sharp lower and upper dimension estimates. There are however some notable exceptions. In particular, we have included in the book a description of all the preeminent results concerning lower and upper dimension estimates, both for repellers and for hyperbolic sets. The nonadditive thermodynamic formalism plays not only a unifying role but also allows one to consider much more general classes of invariant sets. This includes repellers and hyperbolic sets for maps that are not differentiable.

When more complete geometric information is available, one can often obtain sharper estimates for the dimension or even compute its value. On the other hand, this often requires a more elaborate approach, starting essentially with the seminal work of Douady and Oesterlé in [49], who devised an approach to cover the invariant set in a more optimal manner. Incidentally, sharp lower dimension estimates are in general more difficult to obtain than sharp upper dimension estimates. Moreover, in some cases these estimates are either unknown or are only known to occur for almost *all* parameters in some specific classes of invariant sets of nonconformal maps. For completeness, we also give in the book a sufficiently broad panorama of the existing results concerning dimension estimates for repellers of smooth dynamical systems, with emphasis on the relation to the thermodynamic formalism. Among other topics, we consider self-affine repellers, their nonlinear generalizations, and repellers of nonuniformly expanding maps. In particular, Falconer [55, 58] studied a class of limit sets obtained from the composition of affine transformations that are not necessarily conformal.

1.2.2 Subadditive formalism and entropy spectra

We consider in this section the subadditive version of the thermodynamic formalism. We recall that a sequence $\Phi = (\varphi_n)_{n\in\mathbb{N}}$ is said to be *subadditive* if there is a constant $C > 0$ such that

$$\varphi_{n+m} \leq C + \varphi_n + \varphi_m \circ f^n$$

for every $n, m \in \mathbb{N}$. Among the main motivations for the subadditive thermodynamic formalism is the lack of a nonadditive theory of equilibrium measures.

The nonadditive thermodynamic formalism also includes a variational principle for the topological pressure but with a restrictive assumption on the sequence Φ. Namely, consider a sequence of continuous functions $\varphi_n \colon X \to \mathbb{R}$, and assume that there is a continuous function $\varphi \colon X \to \mathbb{R}$ such that

$$\varphi_{n+1} - \varphi_n \circ f \to \varphi \quad \text{uniformly when} \quad n \to \infty. \tag{1.3}$$

Then the nonadditive topological pressure of the sequence $\Phi = (\varphi_n)_{n\in\mathbb{N}}$ with respect to the map f satisfies the variational principle

$$P(\Phi) = \sup_{\mu} \left(h_\mu(f) + \int_X \varphi \, d\mu \right), \tag{1.4}$$

where the supremum is taken over all f-invariant probability measures μ in X. We notice that the classical variational principle for the topological pressure in (1.1) is a particular case of the variational principle in (1.4). Nevertheless, condition (1.3) is a strong requirement, certainly also caused by considering arbitrary sequences.

On the other hand, it is well-known that equilibrium and Gibbs measures play a prominent role in the dimension theory and in the multifractal analysis of dynamical systems. These often provide natural measures sitting on the corresponding invariant sets, that at the same time carry some "dynamical" information (we note that both equilibrium and Gibbs measures depend on the dynamics). For example, they can be measures of full dimension or measures of full entropy. It is sometimes possible to develop the dimension theory or the multifractal analysis of a given dynamics without a variational principle for the topological pressure, and thus without the possibility of looking for equilibrium and Gibbs measures, but the corresponding proofs tend to be much more technical. Moreover, the theory tends to be less rich, although it may be applicable to more general classes of maps and potentials. Overall, it would be desirable to continue using equilibrium and Gibbs measures even when the classical thermodynamic formalism cannot be used.

This justifies the interest in looking for a more general class of sequences of functions for which it is still possible to establish a variational principle, without further hypotheses, and to develop a corresponding theory of equilibrium measures. Somewhat recently, it was shown by Feng and Huang [66] that a natural class is that of subadditive sequences. In fact, they considered the more general class of asymptotically subadditive sequences (see Definition 7.1.1), and established the variational principle

$$P(\Phi) = \sup_{\mu} \left(h_\mu(f) + \lim_{n\to\infty} \int_X \frac{\varphi_n}{n} \, d\mu \right), \tag{1.5}$$

where the supremum is taken over all f-invariant probability measures μ in X. Identity (1.5) was obtained earlier by Cao, Feng and Huang [42] in the particular case of subadditive sequences, and its generalization to arbitrary asymptotically subadditive sequences follows from a minor modification of their proof. Incidentally, one can show that any sequence satisfying (1.3) is asymptotically subadditive.

Feng and Huang also established the existence of equilibrium measures for continuous transformations with upper semicontinuous entropy, without further hypotheses on the asymptotically subadditive sequence. These are measures μ at which the supremum in (1.5) is attained, that is, they satisfy

$$P(\Phi) = h_\mu(f) + \lim_{n\to\infty} \int_X \frac{\varphi_n}{n} \, d\mu.$$

As an application of these results, one can obtain a detailed multifractal analysis of the entropy spectra of generalized Birkhoff averages of an asymptotically subadditive sequence. More precisely, for an asymptotically subadditive sequence $\Phi = (\varphi_n)_{n\in\mathbb{N}}$ we consider the level sets $E(\alpha)$ composed of the points x such that $\varphi_n(x)/n \to \alpha$ when $n \to \infty$. The associated entropy spectrum $\mathcal{E}$ is obtained from computing the topological entropy of the level sets $E(\alpha)$ as a function of α, and its multifractal analysis corresponds to describe the properties of the function $\mathcal{E}$ in terms of the thermodynamic formalism.

In another direction, again taking advantage of the subadditive thermodynamic formalism, one can also give a detailed description of the dimension of a large class of limits sets of geometric constructions, with more explicit formulas when the associated sequences are subadditive. Roughly speaking, a geometric construction corresponds to the geometric structure provided by the rectangles of any Markov partition of a repeller, although now not necessarily determined by an underlying dynamics. More precisely, geometric constructions are defined in terms of certain decreasing sequences of compact sets, such as the intervals of decreasing size in the construction of the middle-third Cantor set. Moreover, even when one can define naturally an induced map for which the limit set of the geometric construction is an invariant set, this map need not be expanding.

1.2.3 Almost additive formalism and Gibbs measures

The almost additive thermodynamic formalism considers a more specific class of sequences, for which it is possible to construct not only equilibrium measures but also Gibbs measures. We recall that a sequence $\Phi = (\varphi_n)_{n\in\mathbb{N}}$ is said to be *almost additive* if there is a constant $C > 0$ such that

$$-C + \varphi_n + \varphi_m \circ f^n \leq \varphi_{n+m} \leq C + \varphi_n + \varphi_m \circ f^n$$

for every $n, m \in \mathbb{N}$. Clearly, any additive sequence $\varphi_n = \sum_{k=0}^{n-1} \varphi \circ f^k$ is almost additive, but there is a large class of nontrivial examples, in particular related to the study of Lyapunov exponents of nonconformal transformations (see Chapter 11 for details).

More precisely, the main objective of the almost additive thermodynamic formalism developed by Barreira in [6], building on earlier work with Gelfert in [10], is not only to establish a variational principle, but also to discuss the existence and uniqueness of equilibrium and Gibbs measures. The notion of Gibbs measure mimics the corresponding notion in the classical thermodynamic formalism. Among other results, the formalism establishes the uniqueness of equilibrium measures for an almost additive sequence Φ with bounded variation as well as some regularity properties of the topological pressure. In addition, the unique equilibrium and Gibbs measures for a given almost additive sequence coincide and are mixing.

The almost additive thermodynamic formalism allows one to develop a new approach to the multifractal analysis of entropy spectra obtained from the level sets of the Lyapunov exponents, for a class of *nonconformal* repellers. The relation can be described as follows. The Lyapunov exponents are naturally associated to the limits of subadditive sequences, obtained from the norms of some products of matrices. Nevertheless, for the class of nonconformal repellers satisfying a cone condition these sequences of functions are almost additive. In particular, this includes repellers with a strongly unstable foliation and repellers with a dominated splitting (see Chapter 11). We are thus able to apply the almost additive thermodynamic formalism to effect a complete multifractal analysis of the entropy spectra. A priori one could also use the subadditive thermodynamic formalism, but we need Gibbs measures and these are only provided by the almost additive thermodynamic formalism.

Further applications of the almost additive thermodynamic formalism include a conditional variational principle for the spectra of almost additive sequences, and a complete description of the dimension spectra of the generalized Birkhoff averages of an almost additive sequence in a conformal hyperbolic set (we refer to Chapter 12 for details and references). We emphasize that we consider simultaneously averages into the future and into the past. More precisely, the dimension spectra are obtained by computing the Hausdorff dimension of the level sets of the generalized Birkhoff averages both for positive and negative time.

1.3 Contents of the book

In this section we describe systematically the contents of the book. The exposition is divided into four parts:

1. classical thermodynamic formalism;
2. nonadditive thermodynamic formalism, with applications to the dimension of repellers and hyperbolic sets;
3. subadditive thermodynamic formalism, with applications to the dimension of limits sets and the multifractal analysis of entropy spectra;
4. almost additive thermodynamic formalism, with applications to the spectra of Lyapunov exponents and the multifractal analysis of dimension spectra.

The first part is of introductory nature and gives a pragmatic introduction to the classical thermodynamic formalism and its relations to symbolic dynamics. Although everything is proven, we develop the theory only as much as needed for the following chapters. Certainly, a large part of the material is available in other sources, but mostly mixed with other topics. In Chapter 2, we introduce the notion of topological pressure, and after establishing its variational principle, we show that there exist equilibrium measures for any expansive transformation. We also present the characterization of the topological pressure as a Carathéodory dimension, which will be very useful later on. Chapter 3 considers the particular case of symbolic dynamics, which plays an important role in many applications of dynamical systems. After presenting a more explicit formula for the topological pressure with respect to the shift map, we construct equilibrium and Gibbs measures avoiding on purpose Perron–Frobenius operators, and using instead a more elementary approach that is sufficient and in fact convenient for our purposes.

In each of the remaining three parts, we discuss the foundations, main results, and main techniques in the interplay between the particular thermodynamic formalism under consideration (either nonadditive, subadditive, or almost additive), and the dimension theory of dynamical systems. Namely, after an initial chapter in which the core of each thermodynamic formalism is presented in detail, we describe several nontrivial applications of that formalism. The following is a systematic description of each part.

In Part II, we discuss the nonadditive thermodynamic formalism and its applications to the dimension theory of repellers and hyperbolic sets. In Chapter 4, after introducing the notion of nonadditive topological pressure as a Carathéodory dimension, we establish some of its basic properties. We also present nonadditive versions of the variational principle for the topological pressure and of Bowen's equation. As an application, Chapter 5 considers the dimension of repellers, which are invariant sets of a hyperbolic noninvertible dynamics. After describing how Markov partitions can be used to model repellers, we present several applications of the nonadditive thermodynamic formalism to the study of their dimension. This includes lower and upper dimension estimates for a large class of repellers, in particular for maps that need not be differentiable. Chapter 6 is dedicated to the dimension of hyperbolic sets, which are invariant sets of a hyperbolic invertible dynamics. The main aim is to develop to a large extent a corresponding theory to that for repellers in the former chapter.

Part III is dedicated to the subadditive thermodynamic formalism and its applications both to dimension theory and multifractal analysis. We consider in Chapter 7 the particular class of asymptotically subadditive sequences, and we develop the theory in several directions. In particular, we present a variational principle for the topological pressure of an arbitrary asymptotically subadditive sequence, and we establish the existence of equilibrium measures for maps with upper semicontinuous entropy. Chapter 8 is dedicated to the study of limits sets of geometric constructions, from the point of view of the dimension theory of dynamical systems. Our main aim is to describe how the theory for repellers developed in Chapter 5 can be extended to this more general setting, with emphasis on the case when the associated sequences are subadditive. In Chapter 9, for the class of asymptotically subadditive sequences, we describe a multifractal analysis of the entropy spectra of the corresponding generalized Birkhoff averages. We consider the general cases when the Kolmogorov–Sinai entropy is not upper semicontinuous and when the topological pressure is not differentiable. We also consider multidimensional sequences, that is, vectors of asymptotically subadditive sequences.

In Part IV, we discuss the almost additive thermodynamic formalism and its application to multifractal analysis. We consider in Chapter 10 the class of almost additive sequences and we develop to a larger extent the nonadditive thermodynamic formalism in this setting. This includes a discussion of the existence and uniqueness of equilibrium and Gibbs measures, both for repellers and for hyperbolic sets. In order to avoid unnecessary technicalities, we first develop the theory for repellers. We then explain how the proofs of the corresponding results for hyperbolic sets and more generally for continuous maps with upper semicontinuous entropy can be obtained from the proofs for repellers. We also describe some regularity properties of the topological pressure for continuous maps with upper semicontinuous entropy. Chapter 11 considers a class of nonconformal repellers to which one can apply the almost additive thermodynamic formalism developed in the former chapter. Namely, we consider the class of repellers satisfying a cone condition, which includes for example repellers with a strongly unstable foliation and repellers with a dominated splitting. In particular, we describe a multifractal analysis of the entropy spectrum of the Lyapunov exponents of a nonconformal repeller. Further applications to multifractal analysis are described in Chapter 12. In particular, we establish a conditional variational principle for the spectra of an almost additive sequence and we give a complete description of the dimension spectra of the corresponding generalized Birkhoff averages in a conformal hyperbolic set, considering simultaneously averages into the future and into the past. We also consider the general case of multidimensional sequences, that is, vectors of almost additive sequences.

1.4 Basic notions

This section collects in a pragmatic manner all the notions and results from dimension theory and ergodic theory that are needed in the book.

1.4.1 Dimension theory

We introduce in this section the notions of Hausdorff dimension and of lower and upper box dimensions, both for sets and measures. We also introduce the notions of lower and upper pointwise dimensions. We refer to the books [7, 60, 152] for details.

The *diameter* of a set $U \subset \mathbb{R}^m$ is defined by

$$\operatorname{diam} U = \sup\{d(x,y) : x, y \in U\},$$

where d is the distance in $\mathbb{R}^m$, and the *diameter* of a collection $\mathcal{U}$ of subsets of $\mathbb{R}^m$ is defined by

$$\operatorname{diam} \mathcal{U} = \sup\{\operatorname{diam} U : U \in \mathcal{U}\}.$$

Given $Z \subset \mathbb{R}^m$ and $\alpha \in \mathbb{R}$, we define the *α-dimensional Hausdorff measure* of Z by

$$m_H(Z,\alpha) = \lim_{\varepsilon \to 0} \inf_{\mathcal{U}} \sum_{U \in \mathcal{U}} (\operatorname{diam} U)^\alpha, \tag{1.6}$$

where the infimum is taken over all finite or countable covers $\mathcal{U}$ of the set Z with diameter $\operatorname{diam} \mathcal{U} \le \varepsilon$.

Definition 1.4.1. The *Hausdorff dimension* of $Z \subset \mathbb{R}^m$ is defined by

$$\dim_H Z = \inf \left\{ \alpha \in \mathbb{R} : m_H(Z, \alpha) = 0 \right\}.$$

The *lower* and *upper box dimensions* of $Z \subset \mathbb{R}^m$ are defined respectively by

$$\underline{\dim}_B Z = \liminf_{\varepsilon \to 0} \frac{\log N(Z, \varepsilon)}{-\log \varepsilon} \quad \text{and} \quad \overline{\dim}_B Z = \limsup_{\varepsilon \to 0} \frac{\log N(Z, \varepsilon)}{-\log \varepsilon},$$

where $N(Z, \varepsilon)$ denotes the least number of balls of radius ε that are needed to cover the set Z.

One can show that

$$\dim_H Z \le \underline{\dim}_B Z \le \overline{\dim}_B Z. \tag{1.7}$$

Now we introduce corresponding notions for measures. Let μ be a finite measure in $X \subset \mathbb{R}^m$.

Definition 1.4.2. The *Hausdorff dimension* and the *lower* and *upper box dimensions* of μ are defined respectively by

$$\begin{aligned} \dim_H \mu &= \inf\{\dim_H Z : \mu(X \setminus Z) = 0\}, \\ \underline{\dim}_B \mu &= \lim_{\delta \to 0} \inf\{\underline{\dim}_B Z : \mu(Z) \ge \mu(X) - \delta\}, \\ \overline{\dim}_B \mu &= \lim_{\delta \to 0} \inf\{\overline{\dim}_B Z : \mu(Z) \ge \mu(X) - \delta\}. \end{aligned}$$

One can show that

$$\dim_H \mu = \lim_{\delta \to 0} \inf\{\dim_H Z : \mu(Z) \ge \mu(X) - \delta\},$$

and thus, it follows from (1.7) that

$$\dim_H \mu \le \underline{\dim}_B \mu \le \overline{\dim}_B \mu.$$

We also introduce the notions of lower and upper pointwise dimensions.

Definition 1.4.3. The *lower* and *upper pointwise dimensions* of the measure μ at the point $x \in X$ are defined by

$$\underline{d}_\mu(x) = \liminf_{r \to 0} \frac{\log \mu(B(x, r))}{\log r} \quad \text{and} \quad \overline{d}_\mu(x) = \limsup_{r \to 0} \frac{\log \mu(B(x, r))}{\log r}.$$

The following result relates the Hausdorff dimension with the lower pointwise dimension.

Theorem 1.4.4. *The following properties hold:*

1. *if* $\underline{d}_\mu(x) \ge \alpha$ *for* μ*-almost every* $x \in X$*, then* $\dim_H \mu \ge \alpha$*;*

2. *if* $\underline{d}_\mu(x) \le \alpha$ *for every* $x \in Z \subset X$*, then* $\dim_H Z \le \alpha$*;*
3. *we have*

$$\dim_H \mu = \operatorname{ess\,sup}\{\underline{d}_\mu(x) : x \in X\}.$$

We also recall a criterion established by Young in [199] for the coincidence between the Hausdorff and box dimensions of a measure.

Theorem 1.4.5. *If* μ *is a finite measure in* $X \subset \mathbb{R}^m$ *and there exists* $d \ge 0$ *such that*

$$\lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} = d$$

for μ*-almost every* $x \in X$*, then*

$$\dim_H \mu = \underline{\dim}_B \mu = \overline{\dim}_B \mu = d.$$

For any finite measure μ invariant under a $C^{1+\varepsilon}$ diffeomorphism with nonzero Lyapunov exponents almost everywhere, it was shown by Barreira, Pesin and Schmeling in [17] that $\underline{d}_\mu(x) = \overline{d}_\mu(x)$ for μ-almost every x.

1.4.2 Ergodic theory

We recall in this section a few basic notions and results from ergodic theory, including Birkhoff's ergodic theorem, the notion of Kolmogorov–Sinai entropy, and the Shannon–McMillan–Breiman theorem. We refer to the books [108, 128, 195] for details.

We first introduce the notion of invariant measure. Let X be a space with a σ-algebra.

Definition 1.4.6. Given a measurable transformation $f\colon X \to X$, a measure μ in X is said to be f*-invariant* if

$$\mu(f^{-1}A) = \mu(A)$$

for every measurable set $A \subset X$.

The study of the transformations with an invariant measure is the main theme of ergodic theory. We denote by $\mathcal{M}_f$ the set of all f-invariant probability measures in X. A measure $\mu \in \mathcal{M}_f$ is said to be *ergodic* if for any f-invariant measurable set $A \subset X$ (this means that $f^{-1}A = A$) either $\mu(A) = 0$ or $\mu(X \setminus A) = 0$.

The following is a basic result from ergodic theory. We denote by $L^1(X,\mu)$ the space of all measurable functions $\varphi\colon X \to \mathbb{R}$ with $\int_X |\varphi|\, d\mu < \infty$.

Theorem 1.4.7 (Birkhoff's ergodic theorem [30]). *Let* $f\colon X \to X$ *be a measurable transformation. For each* $\mu \in \mathcal{M}_f$ *and* $\varphi \in L^1(X,\mu)$ *the limit*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(x))$$

exists for μ-almost every $x \in X$. If in addition μ is ergodic, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(x)) = \int_X \varphi \, d\mu$$

for μ-almost every $x \in X$.

More generally, we have the following result.

Theorem 1.4.8 (see [128]). *Let $f\colon X \to X$ be a measurable transformation and let $\mu \in \mathcal{M}_f$. For each sequence $(\varphi_n)_{n\in\mathbb{N}} \subset L^1(X,\mu)$ converging μ-almost everywhere and in $L^1(X,\mu)$ to a function $\varphi \in L^1(X,\mu)$, the limit*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi_{n-k} \circ f^k$$

exists μ-almost everywhere and in $L^1(X,\mu)$.

Now we recall the notion of entropy. Given $\mu \in \mathcal{M}_f$, let ξ be a *measurable partition* of X, that is, a finite or countable family of measurable subsets of X such that:

1. $\mu(\bigcup_{C\in\xi} C) = 1$;
2. $\mu(C \cap D) = 0$ for every $C, D \in \xi$ with $C \neq D$.

The *entropy* of the partition ξ with respect to μ is defined by

$$H_\mu(\xi) = -\sum_{C\in\xi} \mu(C) \log \mu(C),$$

with the convention that $0 \log 0 = 0$. One can show that

$$H_\mu(\xi) \leq \log \operatorname{card} \xi. \tag{1.8}$$

Definition 1.4.9. The *Kolmogorov–Sinai entropy* or *metric entropy* of f with respect to a measure $\mu \in \mathcal{M}_f$ is defined by

$$h_\mu(f) = \sup \bigl\{ h_\mu(f,\xi) : H_\mu(\xi) < \infty \bigr\}, \tag{1.9}$$

where

$$h_\mu(f,\xi) = \inf_{n\in\mathbb{N}} \frac{1}{n} H_\mu(\xi_n),$$

and where $\xi_n = \bigvee_{k=0}^{n-1} f^{-k}\xi$ is the measurable partition of X composed of the sets

$$C_{i_1\cdots i_n} = \bigcap_{k=0}^{n-1} f^{-k} C_{i_{k+1}}$$

with $C_{i_1}, \ldots, C_{i_n} \in \xi$.

The notion of metric entropy is due to Kolmogorov [115, 116]. It was extended to arbitrary dynamical systems by Sinai [182], in the form (1.9).

One can show that if $(\xi_n)_{n\in\mathbb{N}}$ is a sequence of measurable partitions such that $\bigcup_{n\in\mathbb{N}} \xi_n$ generates the σ-algebra of X, and ξ_{n+1} is a refinement of ξ_n for each $n \in \mathbb{N}$ (this means that each element of ξ_{n+1} is contained in some element of ξ_n), then

$$h_\mu(f) = \lim_{n\to\infty} h_\mu(f, \xi_n). \tag{1.10}$$

We also have

$$h_\mu(f^k) = k h_\mu(f) \quad \text{for each} \quad k \in \mathbb{N}. \tag{1.11}$$

An alternative definition of metric entropy can be introduced as follows. We first note that if ξ is a measurable partition of X, then for μ-almost every $x \in X$ and each $n \in \mathbb{N}$ there exists a single element $\xi_n(x)$ of ξ_n such that $x \in \xi_n(x)$.

Theorem 1.4.10 (Shannon–McMillan–Breiman). *If $f\colon X \to X$ is a measurable transformation, $\mu \in \mathcal{M}_f$, and ξ is a measurable partition of X, then the limit*

$$h_\mu(f, \xi, x) := \lim_{n\to\infty} -\frac{1}{n} \log \mu(\xi_n(x))$$

exists for μ-almost every $x \in X$. Moreover, the function $x \mapsto h_\mu(f, \xi, x)$ is μ-integrable and

$$h_\mu(f, \xi) = \int_X h_\mu(f, \xi, x)\, d\mu(x).$$

The statement in Theorem 1.4.10 was obtained successively in more general forms by several authors. Shannon [177] considered Markov measures, although the statement was only derived rigorously by Khinchin [113] (see also [114]). McMillan [134] obtained the L^1 convergence, and Breiman [41] obtained the convergence almost everywhere.

It is also convenient to introduce the notion of conditional entropy.

Definition 1.4.11. Given measurable partitions ξ and η of X, we define the *conditional entropy* of ξ with respect to η by

$$H_\mu(\xi|\eta) = -\sum_{C\in\xi, D\in\eta} \mu(C \cap D) \log \frac{\mu(C \cap D)}{\mu(D)}.$$

One can show that $H_\mu(\xi|\eta) = 0$ if and only if η is a *refinement* of ξ, that is, if and only if for every $D \in \eta$ there exists $C \in \xi$ such that $\mu(D \setminus C) = 0$.

Proposition 1.4.12. *If ξ and η are measurable partitions of X, then*

$$H_\mu(\xi \vee \eta) = H_\mu(\eta) + H_\mu(\xi|\eta) \le H_\mu(\eta) + H_\mu(\xi), \tag{1.12}$$

and

$$h_\mu(f, \xi) = h_\mu(f, \eta) + H_\mu(\xi|\eta). \tag{1.13}$$

It follows from (1.12) that $H_\mu(\xi \vee \eta) \geq H_\mu(\eta)$, with equality if and only if η is a refinement of ξ.

It is sometimes possible to compute the entropy of a measurable transformation using a single partition.

Definition 1.4.13. Let $f\colon X \to X$ be a measurable transformation. A measurable partition ξ of X is said to be:

1. a *one-sided generator* (with respect to f) if the sets in $\bigcup_{k\in\mathbb{N}\cup\{0\}} f^{-k}\xi$ generate the σ-algebra of X;
2. a *two-sided generator* (with respect to f) if the sets in $\bigcup_{k\in\mathbb{Z}} f^{-k}\xi$ generate the σ-algebra of X.

When there exists a generator the entropy can be computed as follows.

Theorem 1.4.14 (Kolmogorov–Sinai). *Let $f\colon X \to X$ be a measurable transformation and let $\mu \in \mathcal{M}_f$. Then the following properties hold:*

1. *if ξ is a one-sided generator, then $h_\mu(f) = h_\mu(f, \xi)$;*
2. *if ξ is a two-sided generator and f is invertible μ-almost everywhere, then $h_\mu(f) = h_\mu(f, \xi)$.*

Part I

Classical Thermodynamic Formalism

Chapter 2

Thermodynamic Formalism: Basic Notions

This chapter is an introduction to the classical thermodynamic formalism. Although everything is proven, we develop the theory in a pragmatic manner, only as much as needed for the following chapters. We start by introducing the notion of topological pressure, which includes the topological entropy as a special case. After describing some basic properties of the topological pressure, we establish its variational principle, and we show that there exist equilibrium measures for any expansive transformation. This also serves the purpose of presenting some of the central ideas of the theory without the more involved technicalities in later chapters. Moreover, we present the apparently not so well-known characterization of the topological pressure as a Carathéodory dimension. An elaboration of this approach will be very useful later, in particular to introduce the notion of nonadditive topological pressure in Chapter 4. For further developments of the thermodynamic formalism we refer to the books [108, 109, 149, 166, 195].

2.1 Topological pressure

We first introduce the notion of topological pressure in terms of separated sets. This is the most basic notion of the thermodynamic formalism, and it is a generalization of the notion of topological entropy. It also plays a fundamental role in the dimension theory and in the multifractal analysis of dynamical systems, as we substantially illustrate in later chapters.

Let $f\colon X \to X$ be a continuous transformation of a compact metric space (X, d). For each $n \in \mathbb{N}$ we consider the distance d_n in X defined by

$$d_n(x, y) = \max\left\{d(f^k(x), f^k(y)) : 0 \le k \le n-1\right\}.$$

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Definition 2.1.1. Given $\varepsilon > 0$, a set $E \subset X$ is said to be (n,ε)-*separated* (with respect to f) if $d_n(x,y) > \varepsilon$ for every $x,y \in E$ with $x \neq y$.

We note that since the space X is compact, each (n,ε)-separated set E is finite. The notion of topological pressure can now be introduced as follows.

Definition 2.1.2. The *topological pressure* of a continuous function $\varphi\colon X \to \mathbb{R}$ (with respect to f) is defined by

$$P(\varphi) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \sup_E \sum_{x\in E} \exp \sum_{k=0}^{n-1} \varphi(f^k(x)), \tag{2.1}$$

where the supremum is taken over all (n,ε)-separated sets $E \subset X$.

We note that since the function

$$\varepsilon \mapsto \limsup_{n\to\infty} \frac{1}{n} \log \sup_E \sum_{x\in E} \exp \sum_{k=0}^{n-1} \varphi(f^k(x))$$

is nondecreasing, the limit in (2.1) when $\varepsilon \to 0$ is always well defined. The notion of topological pressure was introduced by Ruelle [164] for expansive transformations (see Definitions 2.4.2 and 2.4.5) and by Walters [194] in the general case.

The following example shows that the topological entropy is a particular case of the topological pressure. We recall that the *topological entropy* of f is given by

$$h(f) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log N(d_n, \varepsilon), \tag{2.2}$$

where $N(d_n,\varepsilon)$ is the maximum number of points in X at a d_n-distance at least ε.

Example 2.1.3. For the function $\varphi = 0$ and any (n,ε)-separated set E, we have

$$\sum_{x\in E} \exp \sum_{k=0}^{n-1} \varphi(f^k(x)) = \sum_{x\in E} 1 = \operatorname{card} E.$$

Therefore,

$$\sup_E \sum_{x\in E} \exp \sum_{k=0}^{n-1} \varphi(f^k(x)) = N(d_n,\varepsilon),$$

which yields

$$P(0) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log N(d_n,\varepsilon) = h(f).$$

The original definition of topological entropy is due to Adler, Konheim and McAndrew [1]. The alternative definition in (2.2) was introduced independently by Bowen [36] and Dinaburg [48].

2.2 Equivalent definitions of pressure

We present in this section several equivalent characterizations of the notion of topological pressure, and in particular a characterization as a Carathéodory dimension due to Pesin and Pitskel' [153]. The latter has in particular the advantages of including the case of noncompact sets, and of relating naturally to the notion of Hausdorff dimension (not surprisingly, since the Hausdorff dimension is already defined in terms of a Charathéodory dimension). We also show that the lim sup in (2.1) can be replaced by a lim inf.

Let again $f\colon X \to X$ be a continuous transformation of a compact metric space (X, d). Let also $\mathcal{U}$ be a finite open cover of X. Given $n \in \mathbb{N}$, we denote by $\mathcal{W}_n(\mathcal{U})$ the collection of vectors $U = (U_1, \dots, U_n)$ with $U_1, \dots, U_n \in \mathcal{U}$. For each $U \in \mathcal{W}_n(\mathcal{U})$ we write $m(U) = n$, and we consider the open set

$$X(U) = \big\{x \in X\colon f^{k-1}(x) \in U_k \text{ for } k = 1, \dots, m(U)\big\}. \tag{2.3}$$

A collection $\Gamma \subset \bigcup_{n\in\mathbb{N}} \mathcal{W}_n(\mathcal{U})$ is said to *cover* a set $Z \subset X$ if $\bigcup_{U\in\Gamma} X(U) \supset Z$.

Given a continuous function $\varphi\colon X \to \mathbb{R}$, for each $n \in \mathbb{N}$ we write

$$\varphi_n = \sum_{k=0}^{n-1} \varphi \circ f^k.$$

Moreover, for each $U \in \mathcal{W}_n(\mathcal{U})$ let

$$\varphi(U) = \begin{cases} \sup_{X(U)} \varphi_n & \text{if } X(U) \neq \varnothing, \\ -\infty & \text{if } X(U) = \varnothing. \end{cases}$$

Given $Z \subset X$ and $\alpha \in \mathbb{R}$, we define

$$M_Z(\alpha, \varphi, \mathcal{U}) = \lim_{n\to\infty} \inf_{\Gamma} \sum_{U\in\Gamma} \exp\big(-\alpha m(U) + \varphi(U)\big),$$

where the infimum is taken over all collections $\Gamma \subset \bigcup_{k\geq n} \mathcal{W}_k(\mathcal{U})$ covering Z, with the convention that $e^{-\infty} = 0$. One can show that the function $\alpha \mapsto M_Z(\alpha, \varphi, \mathcal{U})$ jumps from $+\infty$ to 0 at a unique value of α, which we denote by

$$P_Z(\varphi, \mathcal{U}) = \inf\big\{\alpha \in \mathbb{R}\colon M_Z(\alpha, \varphi, \mathcal{U}) = 0\big\}.$$

Moreover, denoting by $\operatorname{diam} \mathcal{U} = \sup_{U\in\mathcal{U}} \operatorname{diam} U$ the diameter of the cover $\mathcal{U}$, we have the following statement.

Theorem 2.2.1. *The limit*

$$P_Z(\varphi) := \lim_{\operatorname{diam} \mathcal{U}\to 0} P_Z(\varphi, \mathcal{U}) \tag{2.4}$$

exists for each continuous function $\varphi\colon X \to \mathbb{R}$.

Proof. Let $\mathcal{V}$ be a finite open cover of X with diameter smaller than the Lebesgue number of $\mathcal{U}$. Then each element $V_i \in \mathcal{V}$ is contained in some set $U(V_i) \in \mathcal{U}$. We write

$$U(V) = (U(V_1), \dots, U(V_n))$$

for each $V = (V_1, \dots, V_n) \in \mathcal{W}_n(\mathcal{V})$. Clearly, if $\Gamma \subset \bigcup_{k\in\mathbb{N}} \mathcal{W}_k(\mathcal{V})$ covers a set Z, then the collection

$$\{U(V) : V \in \Gamma\} \subset \bigcup_{k\in\mathbb{N}} \mathcal{W}_k(\mathcal{U}) \tag{2.5}$$

also covers Z. Now let

$$\gamma(\mathcal{U}) = \sup\{|\varphi(x) - \varphi(y)| : x, y \in X(U) \text{ for some } U \in \mathcal{U}\}. \tag{2.6}$$

Then

$$\varphi(U(V)) \le \varphi(V) + n\gamma(\mathcal{U})$$

for each $V \in \mathcal{W}_n(\mathcal{V})$, and hence,

$$M_Z(\alpha, \varphi, \mathcal{U}) \le M_Z(\alpha - \gamma(\mathcal{U}), \varphi, \mathcal{V}).$$

Therefore, $P_Z(\varphi, \mathcal{U}) \le P_Z(\varphi, \mathcal{V}) + \gamma(\mathcal{U})$, and

$$P_Z(\varphi, \mathcal{U}) - \gamma(\mathcal{U}) \le \liminf_{\operatorname{diam} \mathcal{V}\to 0} P_Z(\varphi, \mathcal{V}).$$

On the other hand, it follows from the uniform continuity of φ on X that $\gamma(\mathcal{U}) \to 0$ when $\operatorname{diam} \mathcal{U} \to 0$. Therefore,

$$\limsup_{\operatorname{diam} \mathcal{U}\to 0} P_Z(\varphi, \mathcal{U}) \le \liminf_{\operatorname{diam} \mathcal{V}\to 0} P_Z(\varphi, \mathcal{V}).$$

This establishes the existence of the limit in (2.4). □

Since X is compact, it has finite open covers of arbitrarily small diameter. This ensures that we can indeed let $\operatorname{diam} \mathcal{U} \to 0$ in (2.4). We emphasize that in Theorem 2.2.1 the set Z need not be compact neither f-invariant. When $Z = X$ it was shown by Pesin and Pitskel' [153] that the number $P_Z(\varphi)$ coincides with the topological pressure introduced in (2.1).

Theorem 2.2.2. *We have $P_X(\varphi) = P(\varphi)$ for each continuous function $\varphi\colon X \to \mathbb{R}$.*

We shall obtain Theorem 2.2.2 as a consequence of a more general statement in Theorem 2.2.3. To formulate the theorem, given a finite open cover $\mathcal{U}$ of X and $n \in \mathbb{N}$ we define

$$Z_n(\varphi, \mathcal{U}) = \inf_{\Gamma} \sum_{U\in\Gamma} \exp a_U,$$

where the infimum is taken over all collections $\Gamma \in \mathcal{W}_n(\mathcal{U})$ covering X, and where a_U is any given number in the interval $[\inf_{X(U)} \varphi_n, \sup_{X(U)} \varphi_n]$, for each set $U \in \mathcal{W}_n(\mathcal{U})$. Moreover, given $\varepsilon > 0$ and $n \in \mathbb{N}$, we define

$$R_n(\varphi, \varepsilon) = \sup_E \sum_{x \in E} \exp \sum_{k=0}^{n-1} \varphi(f^k(x)),$$

where the supremum is taken over all (n, ε)-separated sets $E \subset X$, and also

$$S_n(\varphi, \varepsilon) = \inf_{\mathcal{V}} \sum_{V \in \mathcal{V}} \exp b_V,$$

where the infimum is taken over all finite open covers $\mathcal{V}$ of X by d_n-balls of radius ε, and where b_V is any given number in the interval $[\inf_V \varphi_n, \sup_V \varphi_n]$, for each set $V \in \mathcal{W}_n(\mathcal{V})$.

Theorem 2.2.3. *For each continuous function $\varphi\colon X \to \mathbb{R}$ we have*

$$\begin{aligned} P_X(\varphi) &= \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log R_n(\varphi, \varepsilon) \\ &= \lim_{\varepsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log R_n(\varphi, \varepsilon) \\ &= \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log S_n(\varphi, \varepsilon) \\ &= \lim_{\varepsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log S_n(\varphi, \varepsilon) \\ &= \lim_{\operatorname{diam} \mathcal{U} \to 0} \limsup_{n \to \infty} \frac{1}{n} \log Z_n(\varphi, \mathcal{U}) \\ &= \lim_{\operatorname{diam} \mathcal{U} \to 0} \liminf_{n \to \infty} \frac{1}{n} \log Z_n(\varphi, \mathcal{U}). \end{aligned} \tag{2.7}$$

Proof. Given a finite open cover $\mathcal{U}$ of X, we consider the function

$$\mathcal{Z}_n(\varphi, \mathcal{U}) = \inf_\Gamma \sum_{U \in \Gamma} \exp \varphi(U),$$

where the infimum is taken over all collections $\Gamma \subset \mathcal{W}_n(\mathcal{U})$ covering X.

Lemma 2.2.4. *The limit*

$$c(\varphi, \mathcal{U}) = \lim_{n \to \infty} \frac{1}{n} \log \mathcal{Z}_n(\varphi, \mathcal{U})$$

is well defined.

Proof of the lemma. Take $m, n \in \mathbb{N}$. Given $\Gamma_m \subset \mathcal{W}_m(\mathcal{U})$ and $\Gamma_n \in \mathcal{W}_n(\mathcal{U})$, we define

$$\Gamma_{m,n} = \big\{UV : U \in \Gamma_m, V \in \Gamma_n\big\} \subset \mathcal{W}_{m+n}(\mathcal{U}).$$

Clearly, if Γ_m and Γ_n cover X, then the collection $\Gamma_{m,n}$ also covers X. Moreover,

$$\varphi(UV) \le \varphi(U) + \varphi(V) \quad \text{for each} \quad UV \in \Gamma_{m,n}. \tag{2.8}$$

It follows from (2.8) that

$$\begin{aligned}\mathcal{Z}_{m+n}(\varphi, \mathcal{U}) &\le \sum_{UV \in \Gamma_{m,n}} \exp \varphi(UV) \\ &\le \sum_{U \in \Gamma_m} \exp(\varphi(U)) \times \sum_{V \in \Gamma_m} \exp \varphi(V),\end{aligned}$$

and hence,

$$\mathcal{Z}_{m+n}(\varphi, \mathcal{U}) \le \mathcal{Z}_m(\varphi, \mathcal{U}) \mathcal{Z}_n(\varphi, \mathcal{U}).$$

This readily implies the statement in the lemma. □

Now we observe that

$$Z_n(\varphi, \mathcal{U}) \le \mathcal{Z}_n(\varphi, \mathcal{U}) \le Z_n(\varphi, \mathcal{U}) e^{n\gamma(\mathcal{U})},$$

with $\gamma(U)$ as in (2.6). Therefore,

$$\frac{1}{n} \log Z_n(\varphi, \mathcal{U}) \le \frac{1}{n} \log \mathcal{Z}_n(\varphi, \mathcal{U}) \le \frac{1}{n} \log Z_n(\varphi, \mathcal{U}) + \gamma(\mathcal{U}),$$

and since $\gamma(\mathcal{U}) \to 0$ when $\operatorname{diam} \mathcal{U} \to 0$, we conclude that

$$\begin{aligned}\lim_{\operatorname{diam} \mathcal{U} \to 0} c(\varphi, \mathcal{U}) &= \lim_{\operatorname{diam} \mathcal{U} \to 0} \limsup_{n \to \infty} \frac{1}{n} \log Z_n(\varphi, \mathcal{U}) \\ &= \lim_{\operatorname{diam} \mathcal{U} \to 0} \liminf_{n \to \infty} \frac{1}{n} \log Z_n(\varphi, \mathcal{U}).\end{aligned} \tag{2.9}$$

Lemma 2.2.5. *We have*

$$\lim_{\operatorname{diam} \mathcal{U} \to 0} c(\varphi, \mathcal{U}) = P_X(\varphi). \tag{2.10}$$

Proof of the lemma. Let $\Gamma \subset \bigcup_{n \in \mathbb{N}} \mathcal{W}_n(\mathcal{U})$ be a collection covering X. Since X is compact, we may assume that Γ is finite, and thus that there exists an integer $q \in \mathbb{N}$ such that $\Gamma \subset \bigcup_{n \le q} \mathcal{W}_n(\mathcal{U})$. Let

$$\Gamma^n = \big\{(U_1, \dots, U_n) : U_i \in \Gamma \text{ for } i = 1, \dots, n\big\}$$

for each $n \in \mathbb{N}$. Clearly, the collection Γ^n covers X for each $n \in \mathbb{N}$. Moreover,

$$\varphi(U) \le \sum_{i=1}^n \varphi(U_i)$$

for each $U = (U_1, \ldots, U_n) \in \Gamma^n$. Given $\alpha \in \mathbb{R}$ and setting

$$N(\Gamma) = \sum_{U \in \Gamma} \exp\big(-\alpha m(U) + \varphi(U)\big), \tag{2.11}$$

we thus obtain

$$N(\Gamma^n) \leq \prod_{i=1}^{n} \sum_{U_i \in \Gamma} \exp\big(-\alpha m(U_i) + \varphi(U_i)\big) = N(\Gamma)^n. \tag{2.12}$$

Given $\alpha > P_X(\varphi, \mathcal{U})$, there exists $m \in \mathbb{N}$ and a collection $\Gamma \subset \bigcup_{n \geq m} \mathcal{W}_n(\mathcal{U})$ covering X such that $N(\Gamma) < 1$. Thus, setting

$$\Gamma^\infty = \big\{U : U \in \Gamma^n \text{ for some } n \in \mathbb{N}\big\}, \tag{2.13}$$

it follows from (2.12) that

$$N(\Gamma^\infty) \leq \sum_{n=1}^{\infty} N(\Gamma^n) \leq \sum_{n=1}^{\infty} N(\Gamma)^n < \infty.$$

Moreover, since Γ covers X, given $p \in \mathbb{N}$ and $x \in X$ there exists $U \in \Gamma^\infty$ such that $x \in X(U)$ and $p \leq m(U) < p+q$. We denote by $\Gamma_p^* \subset \mathcal{W}_p(\mathcal{U})$ the collection of all vectors U^* composed of the first p elements of some of these $U \in \Gamma^\infty$. Clearly, Γ_p^* also covers X. Moreover,

$$\varphi(U^*) \leq \varphi(U) + q\|\varphi\|_\infty,$$

where

$$\|\varphi\|_\infty = \sup\big\{|\varphi(x)| : x \in X\big\},$$

and hence,

$$\begin{aligned} N(\Gamma_p^*) &= \sum_{U^* \in \Gamma_p^*} \exp(-\alpha p + \varphi(U^*)) \\ &\leq \sum_{U \in \Gamma^\infty} \exp\big(-\alpha m(U) + \varphi(U) + q\|\varphi\|_\infty\big) e^{\alpha(m(U)-p)} \\ &\leq N(\Gamma^\infty) \max\{1, e^{\alpha q}\} e^{q\|\varphi\|_\infty} < \infty. \end{aligned}$$

Since

$$N(\Gamma_p^*) = e^{-\alpha p} \sum_{U^* \in \Gamma_p^*} \exp \varphi(U^*) \geq e^{-\alpha p} \mathcal{Z}_p(\varphi, \mathcal{U}),$$

we thus obtain

$$e^{-\alpha p} \mathcal{Z}_p(\varphi, \mathcal{U}) \leq N(\Gamma^\infty) \max\{1, e^{\alpha q}\} e^{q\|\varphi\|_\infty}. \tag{2.14}$$

Moreover, since the right-hand side of (2.14) is independent of p, this yields that

$$c(\varphi, \mathcal{U}) = \lim_{p\to\infty} \frac{1}{p} \log \mathcal{Z}_p(\varphi, \mathcal{U}) \le \alpha,$$

and hence, $c(\varphi, \mathcal{U}) \le P_X(\varphi, \mathcal{U})$. Therefore,

$$\lim_{\operatorname{diam}\mathcal{U}\to 0} c(\varphi, \mathcal{U}) \le P_X(\varphi). \tag{2.15}$$

On the other hand,

$$M_X(\alpha, \varphi, \mathcal{U}) = \lim_{n\to\infty} \inf_{\Gamma} \sum_{U\in\Gamma} \exp\big(-\alpha m(U) + \varphi(U)\big),$$

where the infimum is taken over all collections $\Gamma \subset \bigcup_{k\ge n} \mathcal{W}_k(\mathcal{U})$ covering X, and hence,

$$M_X(\alpha, \varphi, \mathcal{U}) \le \liminf_{n\to\infty} \big(e^{-\alpha n} \mathcal{Z}_n(\varphi, \mathcal{U})\big).$$

For $\alpha < P_X(\varphi, \mathcal{U})$, we have $M_X(\alpha, \varphi, \mathcal{U}) = +\infty$, and thus,

$$e^{-\alpha n} \mathcal{Z}_n(\varphi, \mathcal{U}) \ge 1$$

for all sufficiently large n. This implies that

$$c(\varphi, \mathcal{U}) = \lim_{n\to\infty} \frac{1}{n} \log \mathcal{Z}_n(\varphi, \mathcal{U}) \ge \alpha,$$

and hence, $c(\varphi, \mathcal{U}) \ge P_X(\varphi, \mathcal{U})$. Therefore,

$$\lim_{\operatorname{diam}\mathcal{U}\to 0} c(\varphi, \mathcal{U}) \ge P_X(\varphi),$$

which together with (2.15) yields the desired identity. □

Together with (2.9) the former lemma implies that the two last limits in (2.7) are equal to $P_X(\varphi)$. Now we consider the remaining limits.

Lemma 2.2.6. *We have*

$$\begin{aligned} P_X(\varphi) &= \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log R_n(\varphi, \varepsilon) \\ &= \lim_{\varepsilon\to 0} \liminf_{n\to\infty} \frac{1}{n} \log R_n(\varphi, \varepsilon) \\ &= \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log S_n(\varphi, \varepsilon) \\ &= \lim_{\varepsilon\to 0} \liminf_{n\to\infty} \frac{1}{n} \log S_n(\varphi, \varepsilon). \end{aligned} \tag{2.16}$$

Proof of the lemma. Given $\varepsilon > 0$, let $\mathcal{U}$ be a finite open cover $\mathcal{U}$ of X with diameter at most ε. We note that for each $n \in \mathbb{N}$, any two distinct elements of an (n, ε)-separated set $E \subset X$ are in distinct elements of the open cover

$$\mathcal{U}_n = \{X(U) : U \in \mathcal{W}_n(\mathcal{U})\}.$$

Therefore,

$$R_n(\varphi, \varepsilon) \le \mathcal{Z}_n(\varphi, \mathcal{U}). \tag{2.17}$$

Now let 2δ be a Lebesgue number of the cover $\mathcal{U}$. Take $x \in X$. For each $k = 0, \ldots, n-1$, let us also take $U_{k+1} \in \mathcal{U}$ such that $B(f^k(x), \delta) \subset U_{k+1}$. The d_n-ball $B_n(x, \delta)$ of radius δ centered at x satisfies

$$B_n(x, \delta) = \bigcap_{k=0}^{n-1} f^{-k} B(f^k(x), \delta) \subset X(U),$$

where $U = (U_1, \ldots, U_n)$. Therefore, for each finite open cover $\mathcal{V}$ of X by d_n-balls of radius δ and each $V \in \mathcal{V}$, we have

$$\sup_{x \in V} \varphi_n - \inf_{x \in X} \varphi_n \le n\gamma(\mathcal{U}), \tag{2.18}$$

again with $\gamma(U)$ as in (2.6). Furthermore, each element of V is contained in some element of $\mathcal{U}_n$. This implies that

$$A := \inf_{\Gamma} \sum_{U \in \Gamma} \exp \inf_{x \in X(U)} \varphi_n(x) \le S_n(\varphi, \delta), \tag{2.19}$$

where the first infimum is taken over all collections $\Gamma \subset \mathcal{W}_n(\mathcal{U})$ covering X. Moreover, since the d_n-balls of radius δ centered at the elements of some (n, δ)-separated set E may not cover X, we conclude that

$$B := \inf_{\mathcal{V}} \sum_{V \in \mathcal{V}} \exp \inf_{x \in V} \varphi_n(x) \le R_n(\varphi, \delta), \tag{2.20}$$

where the first infimum is taken over all finite open covers $\mathcal{V}$ of X by d_n-balls of radius δ.

Combining the inequalities (2.17), (2.18), (2.19), and (2.20) we successively obtain

$$\begin{aligned} R_n(\varphi, \varepsilon) &\le \mathcal{Z}_n(\varphi, \mathcal{U}) \le e^{n\varphi(\mathcal{U})} A \\ &\le e^{n\gamma(\mathcal{U})} S_n(\varphi, \delta) \le e^{2n\gamma(\mathcal{U})} B \\ &\le e^{2n\gamma(\mathcal{U})} R_n(\varphi, \delta). \end{aligned} \tag{2.21}$$

Since $\delta \to 0$ when $\varepsilon \to 0$ (recall that the diameter of the cover $\mathcal{U}$ is at most ε), it follows from (2.21) together with (2.10) that

$$\begin{aligned}\lim_{\varepsilon\to 0} \liminf_{n\to\infty} \frac{1}{n} \log R_n(\varphi, \varepsilon) &\le P_X(\varphi) \\ &\le \lim_{\delta\to 0} \liminf_{n\to\infty} \frac{1}{n} \log S_n(\varphi, \delta) \\ &\le \lim_{\delta\to 0} \liminf_{n\to\infty} \frac{1}{n} \log R_n(\varphi, \delta)\end{aligned} \tag{2.22}$$

and similarly,

$$\begin{aligned}\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log R_n(\varphi, \varepsilon) &\le P_X(\varphi) \\ &\le \lim_{\delta\to 0} \limsup_{n\to\infty} \frac{1}{n} \log S_n(\varphi, \delta) \\ &\le \lim_{\delta\to 0} \limsup_{n\to\infty} \frac{1}{n} \log R_n(\varphi, \delta).\end{aligned} \tag{2.23}$$

Identity (2.16) follows now readily from (2.22) and (2.23). □

Combining (2.9), (2.10), and (2.16) we obtain all the identities in (2.7), and this completes the proof of the theorem. □

Our proof of Theorem 2.2.3 is based on arguments in [5] and [39]. We note that Theorem 2.2.2 is a particular case of Theorem 2.2.3, since the first limit in (2.7) is precisely the topological pressure (see the Definition 2.1.2).

2.3 Variational principle

We present in this section the variational principle for the topological pressure, which shows that the pressure of any continuous function φ is the supremum of the quantity $h_\mu(f) + \int_X \varphi\, d\mu$ over all f-invariant probability measures μ in X. Here $h_\mu(f)$ is the Kolmogorov–Sinai entropy of the measure μ. The variational principle was established by Ruelle [164] for expansive transformations (see Definitions 2.4.2 and 2.4.5) and by Walters [194] in the general case.

Theorem 2.3.1 (Variational principle). *If $f\colon X \to X$ is a continuous transformation of a compact metric space, and $\varphi\colon X \to \mathbb{R}$ is a continuous function, then*

$$P(\varphi) = \sup_\mu \left(h_\mu(f) + \int_X \varphi\, d\mu \right), \tag{2.24}$$

where the supremum is taken over all f-invariant probability measures μ in X.

Proof. We first establish a lower bound for the topological pressure. For this, let $\eta = \{C_1, \dots, C_k\}$ be a measurable partition of X. Given $\delta > 0$, for each $i = 1, \dots, k$ let $D_i \subset C_i$ be a compact set such that $\mu(C_i \setminus D_i) < \delta$. We also consider the measurable partition

$$\beta = \{D_0, D_1, \dots, D_k\}, \quad \text{where} \quad D_0 = X \setminus \bigcup_{i=1}^k D_i.$$

It follows from (1.13) that

$$h_\mu(f, \eta) \le h_\mu(f, \beta) + H_\mu(\eta|\beta), \tag{2.25}$$

where $H_\mu(\eta|\beta)$ is the conditional entropy of η with respect to β (see Definition 1.4.11).

Lemma 2.3.2. *We have $H_\mu(\eta|\beta) < 1$ for any sufficiently small δ.*

Proof of the lemma. Since $C_i \cap D_i = D_i$ and $C_i \cap D_j = \varnothing$ for $i = 1, \dots, k$ and $j \ne i$, we obtain

$$\begin{aligned} H_\mu(\eta|\beta) = &-\sum_{i=1}^k \mu(C_i \cap D_i) \log \frac{\mu(C_i \cap D_i)}{\mu(D_i)} \\ &-\sum_{i=1}^k \mu(C_i \cap D_0) \log \frac{\mu(C_i \cap D_0)}{\mu(D_0)} \\ &-\sum_{j=1}^k \sum_{i \ne j} \mu(C_i \cap D_j) \log \frac{\mu(C_i \cap D_j)}{\mu(D_j)} \\ = &-\sum_{i=1}^k \mu(C_i \cap D_0) \log \frac{\mu(C_i \cap D_0)}{\mu(D_0)}. \end{aligned} \tag{2.26}$$

Moreover, since $\mu(C_i \cap D_0) \to 0$ when $\delta \to 0$, it follows from the identities in (2.26) that $H_\mu(\eta|\beta) \to 0$ when $\delta \to 0$. □

By Lemma 2.3.2, it follows from (2.25) that

$$h_\mu(f, \eta) < h_\mu(f, \beta) + 1, \tag{2.27}$$

for any sufficiently small δ. Now let

$$\Delta = \inf \bigl\{ d(x, y) : x \in D_i, y \in D_j, i \ne j \bigr\}.$$

Clearly, $\Delta > 0$. Let us take $\varepsilon \in (0, \Delta/2)$ such that

$$|\varphi(x) - \varphi(y)| < 1 \quad \text{whenever} \quad d(x, y) < \varepsilon. \tag{2.28}$$

For each $n \in \mathbb{N}$ and $C \in \beta_n = \bigvee_{j=0}^{n-1} f^{-j}\beta$, there exists $x_C \in \overline{C}$ such that

$$\varphi_n(x_C) = \sup\big\{\varphi_n(x) : x \in C\big\},$$

where $\varphi_n = \sum_{j=0}^{n-1} \varphi\circ f^j$. Now we observe that if E is an $(n, \varepsilon/2)$-separated set with the maximum possible number of elements, then for each C there exists $p_C \in E$ such that $d_n(x_C, p_C) < \varepsilon$. By (2.28), we thus obtain

$$\varphi_n(x_C) \leq \varphi_n(p_C) + n. \tag{2.29}$$

On the other hand, since $\varepsilon < \Delta/2$, for each $x \in E$ and $j = 0, \ldots, n-1$ the point $f^j(x)$ is at most in two elements of β. Therefore,

$$\operatorname{card}\big\{C \in \beta_n : p_C = x\big\} \leq 2^n. \tag{2.30}$$

To proceed with the proof we need the following auxiliary result.

Lemma 2.3.3. *For any numbers $p_i \geq 0$ with $\sum_{i=1}^k p_i = 1$, and $c_i \in \mathbb{R}$ we have*

$$\sum_{i=1}^k p_i(-\log p_i + c_i) \leq \log \sum_{i=1}^k e^{c_i}, \tag{2.31}$$

with equality if and only if

$$p_i = \frac{e^{c_i}}{\sum_{i=1}^k e^{c_i}} \quad \text{for} \quad i = 1, \ldots, k. \tag{2.32}$$

Proof of the lemma. Setting

$$a_i = \frac{e^{c_i}}{\sum_{i=1}^k e^{c_i}} \quad \text{and} \quad x_i = \frac{p_i}{e^{c_i}} \sum_{i=1}^k e^{c_i}$$

for $i = 1, \ldots, k$, we have

$$\sum_{i=1}^k a_i x_i = \sum_{i=1}^k p_i = 1.$$

Since the function $\chi\colon [0,1] \to \mathbb{R}$ defined by

$$\chi(x) = \begin{cases} x\log x & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0 \end{cases} \tag{2.33}$$

is convex, we thus obtain

$$\begin{aligned} 0 &= \chi\left(\sum_{i=1}^k a_i x_i\right) \\ &\le \sum_{i=1}^k a_i \chi(x_i) \\ &= \sum_{i=1}^k \frac{e^{c_i}}{\sum_{i=1}^k e^{c_i}} \cdot \frac{p_i}{e^{c_i}} \sum_{i=1}^k e^{c_i} \log\left(\frac{p_i}{e^{c_i}} \sum_{i=1}^k e^{c_i}\right) \\ &= \sum_{i=1}^k p_i \left(\log p_i - c_i + \log \sum_{i=1}^k e^{c_i}\right) \\ &= \log \sum_{i=1}^k e^{c_i} - \sum_{i=1}^k p_i(-\log p_i + c_i). \end{aligned}$$

This establishes inequality (2.31). Moreover, since the function χ is strictly convex, inequality (2.31) is an identity if and only if $x_1 = \cdots = x_k = d$ for some $d \ge 0$, that is, if and only if

$$p_i = \frac{de^{c_i}}{\sum_{i=1}^k e^{c_i}} \quad \text{for} \quad i = 1, \ldots, k.$$

Summing over i yields $d = 1$, and hence, (2.31) is an identity if and only if (2.32) holds. □

By Lemma 2.3.3 together with (2.29) and (2.30), we obtain

$$\begin{aligned} H_\mu(\beta_n) + \int_X \varphi_n \, d\mu &\le \sum_{C \in \beta_n} \mu(C)\bigl(-\log \mu(C) + \varphi_n(x_C)\bigr) \\ &\le \log \sum_{C \in \beta_n} \exp \varphi_n(x_C) \\ &\le \log \sum_{C \in \beta_n} \exp(\varphi_n(p_C) + n) \\ &\le n + \log\left(2^n \sum_{x \in E} \exp \varphi_n(x)\right), \end{aligned} \tag{2.34}$$

and hence,

$$\begin{aligned} \frac{1}{n} H_\mu(\beta_n) + \int_X \varphi \, d\mu &= \frac{1}{n} H_\mu(\beta_n) + \frac{1}{n} \int_X \varphi_n \, d\mu \\ &\le 1 + \log 2 + \frac{1}{n} \log \sup_E \sum_{x \in E} \exp \varphi_n(x). \end{aligned} \tag{2.35}$$

Together with (2.27), this implies that

$$\begin{aligned} h_\mu(f,\eta) + \int_X \varphi\,d\mu &< h_\mu(f,\beta) + 1 + \int_X \varphi\,d\mu \\ &\le 2 + \log 2 + \limsup_{n\to\infty} \frac{1}{n} \log \sup_E \sum_{x\in E} \exp\varphi_n(x), \end{aligned} \tag{2.36}$$

and letting $\varepsilon \to 0$ yields

$$\begin{aligned} h_\mu(f) + \int_X \varphi\,d\mu &= \sup_\eta \left(h_\mu(f,\eta) + \int_X \varphi\,d\mu \right) \\ &\le 2 + \log 2 + p(f,\varphi), \end{aligned} \tag{2.37}$$

where $p(f,\varphi)$ denotes the topological pressure of φ with respect to f. Now we observe that

$$p(f^m,\varphi_m) = mp(f,\varphi) = mP(\varphi)$$

for each $m \in \mathbb{N}$. Therefore, replacing f by f^m and φ by φ_m in (2.37), it follows from (1.11) that

$$\begin{aligned} h_\mu(f) + \int_X \varphi\,d\mu &= \frac{1}{m} \left(h_\mu(f^m) + \int_X \varphi_m\,d\mu \right) \\ &\le \frac{1}{m} \big(2 + \log 2 + mP(\varphi) \big) \\ &= \frac{2+\log 2}{m} + P(\varphi). \end{aligned}$$

Finally, letting $m \to \infty$ yields

$$h_\mu(f) + \int_X \varphi\,d\mu \le P(\varphi),$$

and hence,

$$\sup_\mu \left(h_\mu(f) + \int_X \varphi\,d\mu \right) \le P(\varphi).$$

Now we establish the reverse inequality. For this, given $\varepsilon > 0$, for each $n \in \mathbb{N}$ we consider a set $E_n \subset X$ of points at a d_n-distance at least ε such that

$$\log \sum_{x\in E_n} \exp\varphi_n(x) > \left(\log \sup_E \sum_{x\in E} \exp\varphi_n(x) \right) - 1, \tag{2.38}$$

where the supremum is taken over all (n,ε)-separated sets. We then define probability measures

$$\nu_n = \frac{\sum_{x\in E_n} e^{\varphi_n(x)} \delta_x}{\sum_{x\in E_n} e^{\varphi_n(x)}}, \tag{2.39}$$

where δ_x is the delta-measure at x, and

$$\mu_n = \frac{1}{n}\sum_{i=0}^{n-1} f_*^i \nu_n, \tag{2.40}$$

where f_* is defined by $(f_*\mu)(A) = \mu(f^{-1}A)$. Now let $(k_n)_{n\in\mathbb{N}} \subset \mathbb{N}$ be a sequence such that

$$\lim_{n\to\infty} \frac{1}{k_n} \log \sum_{x\in E_{k_n}} \exp \varphi_{k_n}(x) = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{x\in E_n} \exp \varphi_n(x), \tag{2.41}$$

and without loss of generality let us also assume that the sequence of measures $(\mu_{k_n})_{n\in\mathbb{N}}$ converges to some measure μ. One can easily verify that μ is an f-invariant probability measure.

We also consider a measurable partition ξ of X such that $\operatorname{diam} C < \varepsilon$ and $\mu(\partial C) = 0$ for each $C \in \xi$. It can be constructed as follows. Let $\{B_1, \ldots, B_p\}$ be an open cover of X by balls of radius less than $\varepsilon/2$ such that $\mu(\partial B_i) = 0$ for $i = 1, \ldots, p$. It always exists since for each $x \in X$ there are at most countably many values of $r > 0$ such that $\mu(\partial B(x, r)) > 0$. We define a measurable partition $\xi = \{C_1, \ldots, C_p\}$ by

$$C_1 = \overline{B_1} \quad \text{and} \quad C_i = \overline{B_i} \setminus \bigcup_{j=1}^{i-1} \overline{B_j} \quad \text{for} \quad i = 2, \ldots, p.$$

Then $\operatorname{diam} C_i < \varepsilon$ and $\mu(\partial C_i) = 0$ for each i, since $\partial C_i \subset \bigcup_{j=1}^{p} \partial B_j$.

Now we write

$$E_n = \{x_1, \ldots, x_k\}, \quad p_i = \nu_n(\{x_i\}), \quad \text{and} \quad c_i = \varphi_n(x_i)$$

for $i = 1, \ldots, k$. We note that

$$p_i = \frac{e^{\varphi_n(x_i)}}{\sum_{x\in E_n} e^{\varphi_n(x)}} = \frac{e^{c_i}}{\sum_{i=1}^{k} e^{c_i}}.$$

By Lemma 2.3.3, this ensures that (2.31) becomes an identity, and hence, writing $\xi_n = \bigvee_{j=0}^{n-1} f^{-j}\xi$, we obtain

$$\begin{aligned} H_{\nu_n}(\xi_n) + n\int_X \varphi\, d\mu_n &= H_{\nu_n}(\xi_n) + \int_X \varphi_n\, d\nu_n \\ &= \sum_{x\in E_n} \nu_n(\{x\})\big(-\log \nu_n(\{x\}) + \varphi_n(x)\big) \\ &= \log \sum_{x\in E_n} \exp \varphi_n(x). \end{aligned} \tag{2.42}$$

Now given $m, n \in \mathbb{N}$ we write $n = qm + r$, where $q \geq 0$ and $0 \leq r < m$. We have

$$\xi_n = \xi_{qm+r} = \bigvee_{j=0}^{q-1} f^{-jm}\xi_m \vee \bigvee_{j=qm}^{qm+r-1} f^{-j}\xi,$$

and thus, for $i = 0, \ldots, m-1$ the partition

$$\eta = \bigvee_{j=0}^{q-1} f^{-jm-i}\xi_m \vee \left(\bigvee_{j=qm}^{qm+r-1} f^{-j}\xi \vee \xi_i \right)$$

is a refinement of ξ_n. Since

$$\operatorname{card}\left(\bigvee_{j=qm}^{qm+r-1} f^{-j}\xi \vee \xi_i \right) \leq (\operatorname{card}\xi)^{2m},$$

it follows from (1.8) and (1.12) that

$$H_{\nu_n}(\xi_n) \leq H_{\nu_n}(\eta) \leq \sum_{j=0}^{q-1} H_{\nu_n}(f^{-jm-i}\xi_m) + 2m \log \operatorname{card}\xi. \tag{2.43}$$

On the other hand, since the function χ in (2.33) is convex, we obtain

$$\begin{aligned} \frac{1}{n}\sum_{i=0}^{m-1}\sum_{j=0}^{q-1} H_{\nu_n}(f^{-jm-i}\xi_m) &\leq \frac{1}{n}\sum_{l=0}^{n-1} H_{\nu_n}(f^{-l}\xi_m) \\ &= -\sum_{A\in\xi_m}\sum_{l=0}^{m-1}\frac{1}{n}\chi\big(\nu_n(f^{-l}A)\big) \\ &\leq -\sum_{A\in\xi_m}\chi\left(\sum_{l=0}^{n-1}\frac{1}{n}\nu_n(f^{-l}A)\right) \\ &= -\sum_{A\in\xi_m}\chi(\mu_n(A)) = H_{\mu_n}(\xi_m). \end{aligned} \tag{2.44}$$

By (2.42), we have

$$\begin{aligned} \frac{m}{n}\log\sum_{x\in E_n}\exp\varphi_n(x) &= \frac{m}{n}H_{\nu_n}(\xi_n) + m\int_X \varphi\,d\mu_n \\ &= \frac{1}{n}\sum_{i=0}^{m-1} H_{\nu_n}(\xi_n) + m\int_X \varphi\,d\mu_n, \end{aligned}$$

and hence, by (2.43) and (2.44),

$$\begin{aligned}
&\frac{m}{n}\log\sum_{x\in E_n}\exp\varphi_n(x)\\
&\le \frac{1}{n}\sum_{i=0}^{m-1}\sum_{j=0}^{q-1}H_{\nu_n}(f^{-jm-i}\xi_m)+\frac{2m^2}{n}\log\operatorname{card}\xi+m\int_X\varphi\,d\mu_n \qquad (2.45)\\
&\le H_{\mu_n}(\xi_m)+\frac{2m^2}{n}\log\operatorname{card}\xi+m\int_X\varphi\,d\mu_n.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{k_n}\log\sum_{x\in E_{k_n}}\exp\varphi_{k_n}(x)\\
&\le \lim_{n\to\infty}\left(\frac{1}{m}H_{\mu_{k_n}}(\xi_m)+\frac{2m}{k_n}\log\operatorname{card}\xi+\int_X\varphi\,d\mu_{k_n}\right) \qquad (2.46)\\
&=\frac{1}{m}H_\mu(\xi_m)+\int_X\varphi\,d\mu.
\end{aligned}$$

To verify that indeed $H_{\mu_{k_n}}(\xi_m)\to H_\mu(\xi_m)$ when $n\to\infty$, let $A\subset X$ be a measurable set with $\mu(\partial A)=0$. Let also $\psi_l\colon X\to\mathbb{R}_0^+$ be a sequence of continuous functions decreasing to $\chi_{\overline{A}}$ when $l\to\infty$. Since $(\mu_{k_n})_{n\in\mathbb{N}}$ converges to μ, we have

$$\limsup_{n\to\infty}\mu_{k_n}(\overline{A})\le\limsup_{n\to\infty}\int_X\psi_l\,d\mu_{k_n}=\int_X\psi_l\,d\mu\to\mu(\overline{A})$$

when $l\to\infty$. Therefore, since $\mu(\partial A)=0$ we obtain

$$\limsup_{n\to\infty}\mu_{k_n}(A)\le\limsup_{n\to\infty}\mu_{k_n}(\overline{A})\le\mu(\overline{A})=\mu(A). \qquad (2.47)$$

Similarly, since $\partial(X\setminus A)=\partial A$ we also have

$$\limsup_{n\to\infty}\mu_{k_n}(X\setminus A)\le\mu(X\setminus A),$$

which yields

$$\liminf_{n\to\infty}\mu_{k_n}(A)\ge\mu(A).$$

Together with (2.47) this implies that

$$\lim_{n\to\infty}\mu_{k_n}(A)=\mu(A),$$

and hence,

$$\lim_{n\to\infty}H_{\mu_{k_n}}(\xi_m)=H_\mu(\xi_m).$$

Letting $m \to \infty$ in (2.46) we finally obtain

$$\lim_{n\to\infty} \frac{1}{k_n} \log \sum_{x\in E_{k_n}} \exp \varphi_{k_n}(x) \le h_\mu(f,\xi) + \int_X \varphi\, d\mu$$
$$\le h_\mu(f) + \int_X \varphi\, d\mu,$$

and hence, by (2.41),

$$\limsup_{n\to\infty} \frac{1}{n} \log \sum_{x\in E_n} \exp \varphi_n(x) \le \sup_\nu \left(h_\nu(f) + \int_X \varphi\, d\nu \right),$$

where the supremum is taken over all f-invariant probability measures ν in X. By (2.38), letting $\varepsilon \to 0$ yields

$$P(\varphi) \le \sup_\nu \left(h_\nu(f) + \int_X \varphi\, d\nu \right),$$

and this completes the proof of the theorem. □

We note that identity (2.24) can also be used as an alternative definition for the topological pressure. Our proof of Theorem 2.3.1 is based on [195], which follows the simpler proof of Misiurewicz in [136].

2.4 Equilibrium measures

We consider in this section the class of invariant measures at which the supremum in (2.24) is attained, the so-called equilibrium measures. These play an important role in the dimension theory and in the multifractal analysis of dynamical systems, where they often occur for example as measures of full entropy or measures of full dimension. In particular, we show that any expansive transformation has equilibrium measures.

Let $f\colon X \to X$ be a continuous transformation of a compact metric space.

Definition 2.4.1. Given a continuous function $\varphi\colon X \to \mathbb{R}$, an f-invariant probability measure μ in X is called an *equilibrium measure* for φ (with respect to f) if

$$P(\varphi) = h_\mu(f) + \int_X \varphi\, d\mu.$$

Now we consider a particular class of transformations.

Definition 2.4.2. A transformation $f\colon X \to X$ is said to be *one-sided expansive* if there exists $\varepsilon > 0$ such that $x = y$ whenever

$$d(f^n(x), f^n(y)) < \varepsilon \quad \text{for every} \quad n \in \mathbb{N} \cup \{0\}. \tag{2.48}$$

The following statement establishes the existence of equilibrium measures for one-sided expansive continuous transformations.

Theorem 2.4.3. *If $f\colon X \to X$ is a one-sided expansive continuous transformation of a compact metric space, then any continuous function $\varphi\colon X \to \mathbb{R}$ has at least one equilibrium measure.*

Proof. Let η be a finite measurable partition of X such that $\operatorname{diam}\eta < \varepsilon$, with ε as in Definition (2.4.2). We first show that the partitions $\eta_n = \bigvee_{k=0}^{n-1} f^{-k}\eta$ satisfy

$$\operatorname{diam}\eta_n \to 0 \quad \text{when} \quad n \to \infty. \tag{2.49}$$

Otherwise, there would exist $\delta > 0$, an increasing sequence $(n_p)_{p\in\mathbb{N}} \subset \mathbb{N}$, and points x_p and y_p for each $p \in \mathbb{N}$, such that

$$d(x_p, y_p) \geq \delta \quad \text{and} \quad x_p, y_p \in \bigcap_{k=0}^{n_p-1} f^{-k}C_{pk}$$

for some sets $C_{pk} \in \eta$. Since X is compact, we can also assume that $x_p \to x$ and $y_p \to y$ when $p \to \infty$, for some points $x, y \in X$. Clearly, $d(x,y) \geq \delta$. Since η is finite, for each k there are infinitely many sets C_{pk} coinciding, say with some set $D_k \in \eta$. Therefore, $x_p, y_p \in f^{-k}D_k$ for infinitely many integers p, and hence, $x, y \in f^{-k}\overline{D_k}$. This shows that (2.48) holds, and since f is one-sided expansive, we conclude that $x = y$. But this contradicts the inequality $d(x,y) \geq \delta$. We have thus established (2.49). This implies that η is a one-sided generator (see Definition 1.4.13), and hence, it follows from Theorem 1.4.14 that

$$h_\mu(f) = h_\mu(f, \eta). \tag{2.50}$$

The following step is to show that the transformation $\mu \mapsto h_\mu(f)$ is upper semicontinuous in the set $\mathcal{M}_f$ of all f-invariant probability measures in X. This means that given a measure $\mu \in \mathcal{M}_f$ and $\delta > 0$, we have $h_\nu(f) < h_\mu(f) + \delta$ for any measure $\nu \in \mathcal{M}_f$ in some open neighborhood of μ. Here we are considering the distance d in $\mathcal{M}_f$ given by

$$d(\mu, \nu) = \sum_{n=1}^{\infty} \frac{1}{2^n} \left| \int_X \varphi_n \, d\mu - \int_X \varphi_n \, d\nu \right|, \tag{2.51}$$

where $\varphi_n\colon X \to \mathbb{R}$ are fixed continuous functions whose union has closure equal to the ball

$$\bigl\{\varphi\colon X \to \mathbb{R} \text{ continuous} : \|\varphi\|_\infty \leq 1\bigr\}.$$

Lemma 2.4.4. *The transformation $\mu \mapsto h_\mu(f)$ is upper semicontinuous.*

Proof of the lemma. Take $\mu \in \mathcal{M}_f$ and let $\xi = \{C_1, \ldots, C_k\}$ be a measurable partition of X with $\operatorname{diam}\xi < \varepsilon$. Given $\delta > 0$, let us take $n \in \mathbb{N}$ such that

$$\frac{1}{n} H_\mu(\xi_n) < h_\mu(f) + \delta, \tag{2.52}$$

where $\xi_n = \bigvee_{j=0}^{n-1} f^{-j}\xi$. Given $\alpha > 0$, for each $i_1, \ldots, i_n \in \{1, \ldots, k\}$ let $K_{i_1\cdots i_n} \subset \bigcap_{j=0}^{n-1} f^{-j} C_{i_{j+1}}$ be a compact set such that

$$\mu\left(\bigcap_{j=0}^{n-1} f^{-j} C_{i_{j+1}} \setminus K_{i_1\cdots i_n}\right) < \alpha. \tag{2.53}$$

Now we consider the sets

$$E_i := \bigcup_{j=0}^{n-1} \bigcup_{i_j=i} f^j(K_{i_1\cdots i_n}) \subset C_i,$$

for $i = 1, \ldots, k$. Since these are pairwise disjoint compact sets, there exists a measurable partition $\eta = \{D_1, \ldots, D_k\}$ of X with $\operatorname{diam} \eta < \varepsilon$ such that $E_i \subset \operatorname{int} D_i$ for $i = 1, \ldots, k$. Clearly,

$$K_{i_1\cdots i_n} \subset \operatorname{int} \bigcap_{j=0}^{n-1} f^{-j} D_{i_{j+1}}.$$

By Urysohn's lemma, for each $i_1, \ldots, i_n \in \{1, \ldots, k\}$ there exists a continuous function $\varphi_{i_1\cdots i_n} \colon X \to [0,1]$ that is 0 on $X \setminus \operatorname{int} \bigcap_{j=0}^{n-1} f^{-j} D_{i_{j+1}}$ and 1 on $K_{i_1\cdots i_n}$. Now we consider the set $V_{i_1\cdots i_n}$ of all measures $\nu \in \mathcal{M}_f$ such that

$$\left| \int_X \varphi_{i_1\cdots i_n} \, d\nu - \int_X \varphi_{i_1\cdots i_n} \, d\mu \right| < \alpha.$$

We note that $V_{i_1\cdots i_n}$ is an open neighborhood of μ_i, with respect to the distance d in (2.51). Then

$$\begin{aligned}
\nu\left(\bigcap_{j=0}^{n-1} f^{-j} D_{i_{j+1}}\right) &\geq \int_X \varphi_{i_1\cdots i_n} \, d\nu \\
&> \int_X \varphi_{i_1\cdots i_n} \, d\mu - \alpha \\
&\geq \mu(K_{i_1\cdots i_n}) - \alpha.
\end{aligned}$$

By (2.53), this implies that

$$\mu\left(\bigcap_{j=0}^{n-1} f^{-j} C_{i_{j+1}}\right) - \nu\left(\bigcap_{j=0}^{n-1} f^{-j} D_{i_{j+1}}\right) < 2\alpha. \tag{2.54}$$

Now let $V = \bigcap_{i_1\cdots i_n} V_{i_1\cdots i_n}$. For each $\nu \in U$ and $i_1, \ldots, i_n \in \{1, \ldots, k\}$, since

$$\sum_{l_1\cdots l_n} \nu\left(\bigcap_{j=0}^{n-1} f^{-j} D_{l_{j+1}}\right) = \sum_{l_1\cdots l_n} \mu\left(\bigcap_{j=0}^{n-1} f^{-j} C_{l_{j+1}}\right) = 1,$$

we have

$$\begin{aligned}
&\nu\Bigg(\bigcap_{j=0}^{n-1} f^{-j}D_{i_{j+1}}\Bigg) - \mu\Bigg(\bigcap_{j=0}^{n-1} f^{-j}C_{i_{j+1}}\Bigg)\\
&= \sum_{(l_1\cdots l_n)\neq(i_1\cdots i_n)} \Bigg[\mu\Bigg(\bigcap_{j=0}^{n-1} f^{-j}C_{l_{j+1}}\Bigg) - \nu\Bigg(\bigcap_{j=0}^{n-1} f^{-j}D_{l_{j+1}}\Bigg)\Bigg] \leq 2\alpha k^n.
\end{aligned}$$

Together with (2.54) this implies that

$$\left|\nu\Bigg(\bigcap_{j=0}^{n-1} f^{-j}D_{i_{j+1}}\Bigg) - \mu\Bigg(\bigcap_{j=0}^{n-1} f^{-j}C_{i_{j+1}}\Bigg)\right| \leq 2\alpha k^n.$$

Therefore, provided that α is sufficiently small, we obtain

$$\frac{1}{n}H_\nu(\eta_n) \leq \frac{1}{n}H_\mu(\xi_n) + \delta.$$

By (2.50) and (2.52), we conclude that

$$\begin{aligned}
h_\nu(f) = h_\nu(f,\eta) &\leq \frac{1}{n}H_\nu(\eta_n)\\
&\leq \frac{1}{n}H_\mu(\xi_n) + \delta \leq h_\mu(f) + 2\delta,
\end{aligned}$$

and hence, the transformation $\mu \mapsto h_\mu(f)$ is upper semicontinuous. □

Since the transformation $\mu \mapsto \int_X \varphi\, d\mu$ is continuous for each given continuous function $\varphi\colon X \to \mathbb{R}$, it follows from Lemma 2.4.4 that

$$\mu \mapsto h_\mu(f) + \int_X \varphi\, d\mu \tag{2.55}$$

is upper semicontinuous. Since an upper semicontinuous function has a maximum in any compact set, it follows from the variational principle in Theorem 2.3.1 that each continuous function φ has an equilibrium measure. This completes the proof of the theorem. □

Theorem 2.4.3 is due to Ruelle [164] (for $\varphi = 0$ the statement was first established by Goodman [79]). Our argument for the upper semicontinuity of the entropy in the proof of Theorem 2.4.3 is based on [195].

In the case of invertible transformations one can consider a weaker notion of expansivity.

Definition 2.4.5. An invertible transformation $f\colon X \to X$ is said to be *two-sided expansive* if there exists $\varepsilon > 0$ such that $x = y$ whenever

$$d(f^n(x), f^n(y)) < \varepsilon \quad \text{for every} \quad n \in \mathbb{Z}.$$

The following statement establishes the existence of equilibrium measures for two-sided expansive homeomorphisms.

Theorem 2.4.6. *If $f\colon X \to X$ is a two-sided expansive homeomorphism of a compact metric space, then any continuous function $\varphi\colon X \to \mathbb{R}$ has at least one equilibrium measure.*

Proof. Let η be a finite measurable partition of X such that $\operatorname{diam} \eta < \varepsilon$, with ε as in Definition 2.4.5. We can show in a similar manner to that in the proof of Theorem 2.4.3 that the partitions

$$\eta_n' = \bigvee_{k=-n}^{n} f^{-k}\eta$$

satisfy $\operatorname{diam} \eta_n' \to 0$ when $n \to \infty$. This implies that η is a two-sided generator (see Definition 1.4.13), and hence, it follows from Theorem 1.4.14 that (2.50) holds. This allows us to repeat the proof of Lemma 2.4.4 to show that in this new situation the transformation $\mu \mapsto h_\mu(f)$ is also upper semicontinuous. Therefore, as in the proof of Theorem 2.4.3, for each continuous function $\varphi\colon X \to \mathbb{R}$ the transformation in (2.55) is upper semicontinuous, and hence φ has at least one equilibrium measure. $\square$

Chapter 3

The Case of Symbolic Dynamics

We consider in this chapter the particular case of symbolic dynamics, which plays an important role in many applications of dynamical systems. In particular, using Markov partitions one can model repellers and hyperbolic sets by their associated symbolic dynamics (see Chapters 5 and 6) of dynamical systems, in this case given by a topological Markov chain (also called a subshift of a finite type). Although the codings of a repeller or a hyperbolic set need not be invertible (due to the boundaries of the Markov partitions), they still provide sufficient information for the applications in dimension theory and in multifractal analysis of dynamical systems. After presenting a more explicit formula for the topological pressure with respect to the shift map, we construct equilibrium and Gibbs measures avoiding on purpose Perron–Frobenius operators, and using instead a more elementary approach that is sufficient for our purposes. An advantage is that an elaboration of this approach will be very useful later, more precisely in the construction of equilibrium and Gibbs measures in the general context of the nonadditive thermodynamic formalism (see Chapters 10 and 11). We consider both one-sided and two-sided sequences, which correspond respectively to the codings of repellers and of hyperbolic sets.

3.1 Topological pressure

We establish in this section a formula for the topological pressure of a continuous function with respect to the shift map. In particular, we show that two limits in (2.1) can be replaced by a single limit in n, and that the $\limsup$ can be replaced by a limit. We first recall some basic notions of symbolic dynamics, starting with the case of one-sided sequences.

For each $\kappa \in \mathbb{N}$, we consider the set $\Sigma_\kappa^+ = \{1, \ldots, \kappa\}^{\mathbb{N}}$ of one-sided sequences of numbers in $\{1, \ldots, \kappa\}$. Given a sequence $\omega \in \Sigma_\kappa^+$ we write it in the form

$$\omega = \big(i_1(\omega) i_2(\omega) \cdots\big).$$

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Moreover, we define the *shift map* $\sigma\colon \Sigma_\kappa^+ \to \Sigma_\kappa^+$ by $\sigma(i_1 i_2 \cdots) = (i_2 i_3 \cdots)$.

Now we introduce a distance, and thus also a topology in Σ_κ^+. Namely, given $\beta > 1$, for each $\omega, \omega' \in \Sigma_\kappa^+$ we set

$$d_\beta(\omega, \omega') = \begin{cases} \beta^{-n} & \text{if } \omega \neq \omega', \\ 0 & \text{if } \omega = \omega', \end{cases} \tag{3.1}$$

where $n = n(\omega, \omega') \in \mathbb{N}$ is the smallest integer such that $i_n(\omega) \neq i_n(\omega')$. One can easily verify that d_β is a distance in Σ_κ^+, with respect to which Σ_κ^+ is compact and the shift map σ is continuous. We note that all distances d_β induce the same topology in Σ_κ^+.

Given $m \in \mathbb{N}$ and $i_1, \ldots, i_m \in \{1, \ldots, \kappa\}$ we define the *m-cylinder set*

$$C_{i_1 \ldots i_m} = \big\{ (j_1 j_2 \cdots) \in \Sigma_\kappa^+ : j_l = i_l \text{ for } l = 1, \ldots, m \big\}.$$

The σ-algebra generated by these sets coincides with the Borel σ-algebra obtained from the distance d_β.

Now we give a more explicit formula for the topological pressure with respect to the shift map.

Theorem 3.1.1. *For each continuous function $\varphi\colon \Sigma_\kappa^+ \to \mathbb{R}$ we have*

$$P(\varphi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{i_1 \cdots i_n} \exp \sup_{C_{i_1 \cdots i_n}} \sum_{l=0}^{n-1} \varphi \circ \sigma^l. \tag{3.2}$$

Proof. Since the space Σ_κ^+ is compact, the function φ is uniformly continuous, and thus, for each $\delta > 0$ there exists $n \in \mathbb{N}$ such that

$$\sup_{C_{i_1 \cdots i_n}} \varphi - \inf_{C_{i_1 \cdots i_n}} \varphi < \delta$$

for every $i_1, \ldots, i_n \in \{1, \ldots, \kappa\}$. Writing for simplicity $D_n = C_{i_1 \cdots i_n}$, we thus obtain

$$\delta_n := \max_{i_1 \cdots i_n} \left(\sup_{D_n} \varphi - \inf_{D_n} \varphi \right) \to 0$$

when $n \to \infty$. Setting

$$\varphi_n = \sum_{l=0}^{n-1} \varphi \circ \sigma^l,$$

this implies that for each $m, n \in \mathbb{N}$, we have

$$\sup_{D_{m+n-1}} \varphi_n \leq \inf_{D_{m+n-1}} \varphi_n + n\delta_m,$$

and hence,

$$0 \leq \log \sum_{i_1 \cdots i_{m+n-1}} \exp \sup_{D_{m+n-1}} \varphi_n - \log \sum_{i_1 \cdots i_{m+n-1}} \exp \inf_{D_{m+n-1}} \varphi_n \leq n\delta_m. \tag{3.3}$$

On the other hand, given $m,n\in\mathbb{N}$ and an (n,ε)-separated set $E\subset\Sigma_\kappa^+$, we have

$$\sum_{i_1\cdots i_{m+n-1}}\exp\inf_{D_{m+n-1}}\varphi_n\le\sum_{x\in E}\exp\varphi_n(x)\le\sum_{i_1\cdots i_{m+n-1}}\exp\sup_{D_{m+n-1}}\varphi_n,$$

and hence,

$$\begin{aligned}&-\delta_m+\limsup_{n\to\infty}\frac{1}{n}\log\sum_{i_1\cdots i_{m+n-1}}\exp\sup_{D_{m+n-1}}\varphi_n\\&\le\limsup_{n\to\infty}\frac{1}{n}\log\sum_{x\in E}\exp\varphi_n(x)\\&\le\limsup_{n\to\infty}\frac{1}{n}\log\sum_{i_1\cdots i_{m+n-1}}\exp\sup_{D_{m+n-1}}\varphi_n.\end{aligned}\tag{3.4}$$

Letting $\varepsilon\to0$ and $m\to\infty$, it follows from (3.3) and (3.4) that

$$\begin{aligned}P(\varphi)&=\lim_{m\to\infty}\limsup_{n\to\infty}\frac{1}{n}\log\sum_{i_1\cdots i_{m+n-1}}\exp\sup_{D_{m+n-1}}\varphi_n\\&=\lim_{m\to\infty}\limsup_{n\to\infty}\frac{1}{n}\log\sum_{i_1\cdots i_{m+n-1}}\exp\inf_{D_{m+n-1}}\varphi_n.\end{aligned}\tag{3.5}$$

The following step is to relate the limits in (3.5) to the quantity

$$S_n=\sum_{i_1\cdots i_n}\exp\sup_{C_{i_1\cdots i_n}}\varphi_n.$$

We first observe that

$$\sum_{i_1\cdots i_{m+n-1}}\exp\sup_{D_{m+n-1}}\varphi_n\le\kappa^{m-1}S_n,$$

and

$$\sum_{i_1\cdots i_{m+n-1}}\exp\sup_{D_{m+n-1}}\varphi_n\ge e^{-(m-1)\|\varphi\|_\infty}S_{m+n-1},$$

where

$$\|\varphi\|_\infty=\sup\left\{|\varphi(\omega)|:\omega\in\Sigma_\kappa^+\right\}.$$

This implies that

$$\lim_{m\to\infty}\limsup_{n\to\infty}\frac{1}{n}\log\sum_{i_1\cdots i_{m+n-1}}\exp\sup_{D_{m+n-1}}\varphi_n\le\limsup_{n\to\infty}\frac{1}{n}\log S_n,$$

and

$$\lim_{m\to\infty}\limsup_{n\to\infty}\frac{1}{n}\log\sum_{i_1\cdots i_{m+n-1}}\exp\sup_{D_{m+n-1}}\varphi_n \geq \lim_{m\to\infty}\limsup_{n\to\infty}\frac{1}{n}\log S_{m+n-1}$$
$$=\limsup_{m\to\infty}\frac{1}{n}\log S_n.$$

Therefore,

$$\begin{aligned} P(\varphi) &= \lim_{m\to\infty}\limsup_{n\to\infty}\frac{1}{n}\log\sum_{i_1\cdots i_{m+n-1}}\exp\sup_{D_{m+n-1}}\varphi_n \\ &= \limsup_{n\to\infty}\frac{1}{n}\log S_n. \end{aligned} \tag{3.6}$$

On the other hand, since

$$\begin{aligned} \sup_{C_{i_1\cdots i_{m+n}}}\varphi_{m+n} &\leq \sup_{C_{i_1\cdots i_{m+n}}}\varphi_m + \sup_{C_{i_1\cdots i_{m+n}}}(\varphi_n\circ\sigma^m) \\ &\leq \sup_{C_{i_1\cdots i_m}}\varphi_m + \sup_{C_{i_{m+1}\cdots i_{m+n}}}\varphi_n, \end{aligned}$$

we have

$$\begin{aligned} S_{m+n} &\leq \sum_{i_1\cdots i_{m+n}}\exp\left(\sup_{C_{i_1\cdots i_m}}\varphi_m + \sup_{C_{i_{m+1}\cdots i_{m+n}}}\varphi_n\right) \\ &= \sum_{i_1\cdots i_m}\exp\sup_{C_{i_1\cdots i_m}}\varphi_m\sum_{i_{m+1}\cdots i_{m+n}}\exp\sup_{C_{i_{m+1}\cdots i_{m+n}}}\varphi_n \\ &= S_m S_n. \end{aligned}$$

This implies that the limit

$$\lim_{n\to\infty}\frac{1}{n}\log S_n$$

exists. Identity (3.2) follows now readily from (3.5) and (3.6). □

Although the formula for the topological pressure in (3.2) has often been used in the literature, to the best of our knowledge no explicit proof has been written before. Moreover, even though the argument can essentially be considered a nontrivial exercise, it is sufficiently involved to deserve being written explicitly.

The following is a simple application of Theorem 3.1.1.

Example 3.1.2. Given numbers $\lambda_1,\ldots,\lambda_\kappa > 0$, we consider the continuous function $\varphi\colon \Sigma_\kappa^+ \to \mathbb{R}$ defined by

$$\varphi(i_1 i_2\cdots) = \log\lambda_{i_1}.$$

It follows from (3.2) that

$$\begin{aligned} P(\varphi) &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{i_1\cdots i_n} \exp \sum_{l=1}^{n} \log \lambda_{i_l} \\ &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{i_1\cdots i_n} \prod_{l=1}^{n} \lambda_{i_l} \\ &= \lim_{n\to\infty} \frac{1}{n} \log \left[\left(\sum_{j=1}^{\kappa} \lambda_j \right)^n \right] = \log \sum_{j=1}^{\kappa} \lambda_j. \end{aligned}$$

3.2 Two-sided sequences

We consider in this section the case of two-sided sequences. After introducing some basic notions of symbolic dynamics, we present a corresponding formula for the topological pressure. The details are identical to those in Section 3.1 and thus are omitted.

For each $\kappa \in \mathbb{N}$, we consider the set $\Sigma_\kappa = \{1, \ldots, \kappa\}^{\mathbb{Z}}$ of two-sided sequences of numbers in $\{1, \ldots, \kappa\}$. Given a sequence $\omega \in \Sigma_\kappa$ we write it in the form

$$\omega = \big(\cdots i_{-1}(\omega) i_0(\omega) i_1(\omega) \cdots \big).$$

Moreover, we define the *shift map* $\sigma\colon \Sigma_\kappa \to \Sigma_\kappa$ by $\sigma(\omega) = \omega'$, where

$$i_n(\omega') = i_{n+1}(\omega) \quad \text{for each} \quad n \in \mathbb{Z}.$$

Now we introduce a distance in Σ_κ. Given $\beta > 1$, for each $\omega, \omega' \in \Sigma_\kappa$ we set

$$d_\beta(\omega, \omega') = \begin{cases} \beta^{-n} & \text{if } \omega \neq \omega', \\ 0 & \text{if } \omega = \omega', \end{cases} \tag{3.7}$$

where $n = n(\omega, \omega') \in \mathbb{N} \cup \{0\}$ is the smallest integer such that $i_n(\omega) \neq i_n(\omega')$ or $i_{-n}(\omega) \neq i_{-n}(\omega')$. One can easily verify that d_β is a distance in Σ_κ, with respect to which Σ_κ is compact and the shift map σ is a homeomorphism. We note that all distances d_β induce the same topology in Σ_κ.

Given $m \in \mathbb{N}$ and $i_{-m}, \ldots, i_m \in \{1, \ldots, \kappa\}$ we define the *cylinder set*

$$C_{i_{-m}\cdots i_m} = \big\{ (\cdots j_0 \cdots) \in \Sigma_\kappa : j_l = i_l \text{ for } l = -m, \ldots, m \big\}.$$

In a similar manner to that for one-sided sequences, the σ-algebra generated by these sets coincides with the Borel σ-algebra obtained from the distance d_β in (3.7).

Slightly modifying the proof of Theorem 3.1.1 we obtain a corresponding formula for the topological pressure.

Theorem 3.2.1. *For each continuous function $\varphi\colon X \to \mathbb{R}$ we have*

$$P(\varphi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{i_1\cdots i_n} \exp \sup_{C_{i_1\cdots i_n}} \sum_{l=0}^{n-1} \varphi \circ \sigma^l.$$

3.3 Equilibrium measures

We consider briefly in this section the problem of the existence of equilibrium measures in the particular case of symbolic dynamics. We recall that an equilibrium measure is an invariant probability measure at which the supremum in the variational principle is attained.

The following results are easy consequences of the expansivity of the shift map.

Theorem 3.3.1. *For the shift map in Σ_κ^+, any continuous function $\varphi\colon \Sigma_\kappa^+ \to \mathbb{R}$ has at least one equilibrium measure.*

Proof. It follows easily from (3.1) that the shift map in Σ_κ^+ is one-sided expansive. The statement is thus an immediate consequence of Theorem 2.4.3. ☐

A similar result holds in the case of two-sided sequences.

Theorem 3.3.2. *For the shift map in Σ_κ, any continuous function $\varphi\colon \Sigma_\kappa \to \mathbb{R}$ has at least one equilibrium measure.*

Proof. It follows easily from (3.7) that the shift map in Σ_κ is two-sided expansive. The statement is thus an immediate consequence of Theorem 2.4.6. ☐

3.4 Gibbs measures

We consider in this section the problem of the existence of Gibbs measures and their relation to equilibrium measures (while an invariant Gibbs measure is always an equilibrium measure, the converse need not be true). In particular, we construct Gibbs measures for any Hölder continuous function on a topologically mixing topological Markov chain. On purpose, we avoid Perron–Frobenius operators, and we use instead a more elementary approach that is sufficient for our purposes. This approach is in fact more convenient for some developments of the theory in later chapters. Although everything is proven, we proceed in a pragmatic manner, only as much as needed for the following chapters.

We first recall the notion of Gibbs measure in the space Σ_κ^+.

Definition 3.4.1. Given a continuous function $\varphi\colon \Sigma_\kappa^+ \to \mathbb{R}$, we say that a probability measure μ in Σ_κ^+ is a *Gibbs measure* for φ if there exists $K \geq 1$ such that

$$K^{-1} \leq \frac{\mu(C_{i_1\cdots i_n})}{\exp\left[-nP(\varphi) + \sum_{l=0}^{n-1} \varphi(\sigma^l(\omega))\right]} \leq K \tag{3.8}$$

for every $(i_1 i_2 \cdots) \in \Sigma_\kappa^+$, $n \in \mathbb{N}$, and $\omega \in C_{i_1\cdots i_n}$.

We show that invariant Gibbs measures are equilibrium measures.

Theorem 3.4.2. *If a probability measure μ in Σ_κ^+ is a σ-invariant Gibbs measure for φ, then it is also an equilibrium measure for φ.*

Proof. By Birkhoff's ergodic theorem (Theorem 1.4.7), it follows from (3.8) that

$$P(\varphi) = \lim_{n\to\infty} -\frac{1}{n}\log\mu(C_{i_1\cdots i_n}) + \lim_{n\to\infty}\frac{1}{n}\sum_{l=0}^{n-1}\varphi(\sigma^l(\omega)) \tag{3.9}$$

for μ-almost every $\omega \in \Sigma_\kappa^+$. Alternatively, the existence μ-almost everywhere, and in $L^1(\Sigma_\kappa^+, \mu)$, of the first limit in (3.9) is a consequence of the Shannon–McMillan–Breiman theorem (Theorem 1.4.10), which also says that

$$\int_{\Sigma_\kappa^+} \lim_{n\to\infty} -\frac{1}{n}\log\mu(C_{i_1\cdots i_n})\,d\mu(\omega) = h_\mu(\sigma), \tag{3.10}$$

where $\omega = (i_1 i_2 \cdots)$. Thus, it follows from (3.9) that

$$\begin{aligned} P(\varphi) &= \int_{\Sigma_\kappa^+} \lim_{n\to\infty} -\frac{1}{n}\log\mu(C_{i_1\cdots i_n})\,d\mu(\omega) \\ &\quad + \int_{\Sigma_\kappa^+} \lim_{n\to\infty}\frac{1}{n}\sum_{l=0}^{n-1}\varphi(\sigma^l(\omega))\,d\mu(\omega) \\ &= h_\mu(\sigma) + \int_{\Sigma_\kappa^+}\varphi\,d\mu, \end{aligned}$$

where the last identity is a consequence of (3.10) and Birkhoff's ergodic theorem. This shows that μ is an equilibrium measure for φ. □

Now we consider the shift map restricted to the particular class of shift-invariant sets in Σ_κ^+ given by the topological Markov chains, and we show that any Hölder continuous function has a σ-invariant Gibbs measure.

We first recall the notion of a topological Markov chain.

Definition 3.4.3. Given a $\kappa\times\kappa$ matrix $A = (a_{ij})$ with $a_{ij} \in \{0,1\}$ for each i and j, we consider the set

$$\Sigma_A^+ = \big\{\omega \in \Sigma_\kappa^+ : a_{i_n(\omega)i_{n+1}(\omega)} = 1 \text{ for every } n \in \mathbb{N}\big\}.$$

Then the restriction $\sigma|\Sigma_A^+ \colon \Sigma_A^+ \to \Sigma_A^+$ is called the *(one-sided) topological Markov chain* or *subshift of finite type* with transition matrix A.

We are now ready to establish the existence of Gibbs measures. As we have already mentioned, instead of using Perron–Frobenius operators, we use instead a more elementary approach.

Theorem 3.4.4. *Assume that A^q has only positive entries for some $q \in \mathbb{N}$. Then for the topological Markov chain $\sigma|\Sigma_A^+$, any Hölder continuous function $\varphi\colon \Sigma_A^+ \to \mathbb{R}$ has at least one σ-invariant Gibbs measure.*

Proof. We first obtain some relations between the numbers

$$a_{i_1\cdots i_n} = \max\left\{\exp\varphi_n(\omega) : \omega \in C_{i_1\cdots i_n}\right\},$$

where $\varphi_n = \sum_{l=0}^{n-1} \varphi \circ \sigma^l$. We also set

$$\alpha_n = \sum_{i_1\cdots i_n} a_{i_1\cdots i_n}.$$

These numbers play a crucial role in the construction of the Gibbs measures.

For each $l > n$ we have

$$a_{i_1\cdots i_n j_1\cdots j_{l-n}} \le a_{i_1\cdots i_n} a_{j_1\cdots j_{l-n}}, \tag{3.11}$$

and hence,

$$\sum_{j_1\cdots j_{l-n}} a_{i_1\cdots i_n j_1\cdots j_{l-n}} \le a_{i_1\cdots i_n} \alpha_{l-n}.$$

Summing over $i_1 \cdots i_n$, we thus obtain

$$\alpha_l \le \alpha_n \alpha_{l-n}. \tag{3.12}$$

On the other hand, given $(i_1 \cdots), (j_1 \cdots) \in \Sigma_A^+$ there exists $(m_1 \cdots) \in \Sigma_A^+$ such that

$$\gamma = (i_1 \cdots i_n m_1 \cdots m_p j_1 \cdots j_{l-p})$$

consists of the first $n+l$ elements of some sequence in Σ_A^+, where $p = q-1$. Hence, for each $\omega \in C_\gamma$ we have

$$a_\gamma \ge \exp\left[\varphi_n(\omega) + \varphi_p(\sigma^n(\omega)) + \varphi_{l-p}(\sigma^{n+p}(\omega))\right]. \tag{3.13}$$

Assuming that the point ω also satisfies the identity

$$\exp\varphi_{l-p}(\sigma^{n+p}(\omega)) = a_{j_1\cdots j_{l-p}},$$

and setting $D = e^{-p\|\varphi\|_\infty}$, it follows from (3.13) that

$$a_\gamma \ge De^{\varphi_n(\omega)+\varphi_{l-p}(\sigma^{n+p}(\omega))} = De^{\varphi_n(\omega)} a_{j_1\cdots j_{l-p}}. \tag{3.14}$$

Since φ is Hölder continuous, for every $\omega, \omega' \in C_{i_1\cdots i_n}$ we have

$$\begin{aligned} |\varphi_n(\omega) - \varphi_n(\omega')| &\le \sum_{l=0}^{n-1} |\varphi(\sigma^l(\omega)) - \varphi(\sigma^l(\omega'))| \\ &\le \sum_{l=0}^{n-1} c\left[d_\beta\left(\sigma^l(\omega), \sigma^l(\omega')\right)\right]^\alpha \\ &\le \sum_{l=0}^{n-1} c\beta^{-\alpha(n+1-l)} \le \frac{c\beta^{-2\alpha}}{1-\beta^{-\alpha}} =: D' < \infty, \end{aligned} \tag{3.15}$$

for some constants $c > 0$ and $\alpha \in (0,1]$. Therefore, by (3.14),

$$a_\gamma \ge De^{-D'} a_{i_1\cdots i_n} a_{j_1\cdots j_{l-p}}, \tag{3.16}$$

and hence,

$$\begin{aligned} \alpha_{n+l} &= \sum_\gamma a_\gamma \\ &\ge De^{-D'} \sum_{i_1\cdots i_n j_1\cdots j_{l-p}} a_{i_1\cdots i_n} a_{j_1\cdots j_{l-p}} \\ &= De^{-D'} \alpha_n \alpha_{l-p}. \end{aligned} \tag{3.17}$$

Finally, we observe that

$$\begin{aligned} \alpha_l = \sum_{i_1\cdots i_l} a_{i_1\cdots i_l} &\le \kappa^p \sum_{i_1\cdots i_{l-p}} a_{i_1\cdots i_l} \\ \le \kappa^p \sum_{i_1\cdots i_{l-p}} a_{i_1\cdots i_{l-p}} e^{p\|\varphi\|_\infty} &= (\kappa e^{\|\varphi\|_\infty})^p \alpha_{l-p}, \end{aligned} \tag{3.18}$$

which together with (3.17) yields

$$\alpha_{n+l} \ge De^{-D'} (\kappa e^{\|\varphi\|_\infty})^{-p} \alpha_n \alpha_l. \tag{3.19}$$

It follows from (3.12) and (3.19) that

$$\begin{aligned} P(\varphi) &= \lim_{n\to\infty} \frac{1}{n} \log \alpha_n \\ &= \inf_{n\in\mathbb{N}} \frac{1}{n} \log \alpha_n \\ &= \sup_{n\in\mathbb{N}} \frac{1}{n} \log(\alpha_n/L), \end{aligned}$$

where

$$L = D^{-1} e^{D'} \left(\kappa e^{\|\varphi\|_\infty}\right)^p = e^{D'} \left(\kappa e^{2\|\varphi\|_\infty}\right)^p.$$

This implies that

$$\log(\alpha_n/L) \le nP(\varphi) \le \log \alpha_n. \tag{3.20}$$

Now for each $n \in \mathbb{N}$ we consider the probability measure ν_n defined in the algebra generated by the sets $C_{i_1\cdots i_n}$ such that

$$\nu_n(C_{i_1\cdots i_n}) = a_{i_1\cdots i_n}/\alpha_n$$

for each $(i_1 i_2 \cdots) \in \Sigma_A^+$. Then

$$\nu_l(C_{i_1\cdots i_n}) = \sum_{j_1\cdots j_{l-n}} \frac{a_{i_1\cdots i_n j_1\cdots j_{l-n}}}{\alpha_l},$$

and by (3.11), (3.15), and (3.19) we obtain

$$\begin{aligned}\nu_l(C_{i_1\cdots i_n}) &\le \frac{a_{i_1\cdots i_n}}{\alpha_l}\sum_{j_1\cdots j_{l-n}} a_{j_1\cdots j_{l-n}}\\ &= \frac{a_{i_1\cdots i_n}}{\alpha_l}\alpha_{l-n}\\ &\le \frac{Le^{D'+\varphi_n(\omega)}}{\alpha_n}\\ &\le Le^{D'}\exp(-nP(\varphi)+\varphi_n(\omega))\end{aligned} \tag{3.21}$$

for every $\omega \in C_{i_1\cdots i_n}$ and $l \in \mathbb{N}$. Similarly, provided that l is sufficiently large, it follows from (3.16), (3.18), and (3.12) that

$$\begin{aligned}\nu_l(C_{i_1\cdots i_n}) &\ge \frac{De^{-D'}a_{i_1\cdots i_n}}{\alpha_l}\sum_{j_{p+1}\cdots j_{l-n}} a_{j_{p+1}\cdots j_{l-n}}\\ &= \frac{De^{-D'}a_{i_1\cdots i_n}\alpha_{l-n-p}}{\alpha_l}\\ &\ge \frac{De^{-D'}e^{\varphi_n(\omega)}}{\alpha_{n+p}}\\ &\ge \frac{De^{-D'}e^{\varphi_n(\omega)}}{(\kappa e^{\|\varphi\|_\infty})^p\alpha_n}\\ &\ge L^{-2}\exp(-nP(\varphi)+\varphi_n(\omega))\end{aligned} \tag{3.22}$$

for every $\omega \in C_{i_1\cdots i_n}$. Now let ν be any sublimit of the sequence of measures $(\nu_l)_{l\in\mathbb{N}}$. It follows readily from (3.21) and (3.22) that

$$L^{-2} \le \frac{\nu(C_{i_1\cdots i_n})}{\exp(-nP(\varphi)+\varphi_n(\omega))} \le L^2$$

for every $n \in \mathbb{N}$ and $\omega \in C_{i_1\cdots i_n}$. Therefore, ν is a Gibbs measure for φ.

Nevertheless, the measure ν need not be σ-invariant. To obtain a σ-invariant Gibbs measure, we consider the sequence of probability measures

$$\mu_n = \frac{1}{n}\sum_{k=0}^{n-1}\nu\circ\sigma^{-k}$$

in Σ_A^+. Since σ is continuous, any sublimit of this sequence is a σ-invariant probability measure in Σ_A^+. Moreover, by (3.20) and (3.21), we have

$$\begin{aligned}\nu(\sigma^{-k}C_{i_1\cdots i_n}) &= \sum_{j_1\cdots j_k}\nu(C_{j_1\cdots j_k i_1\cdots i_n})\\ &\le L^2\sum_{j_1\cdots j_k}\exp\big(-(n+k)P(\varphi)+\varphi_{n+k}(\omega_{j_1\cdots j_k})\big)\end{aligned}$$

for every $k, n \in \mathbb{N}$ and $\omega_{j_1\cdots j_k} \in C_{j_1\cdots j_k i_1\cdots i_n}$. Now we take $\omega \in C_{i_1\cdots i_n}$ and sequences $\omega_{j_1\cdots j_k}$ such that $\sigma^k(\omega_{j_1\cdots j_k}) = \omega$. Then

$$\begin{aligned}
\nu(\sigma^{-k}C_{i_1\cdots i_n}) &\leq L^2 \sum_{j_1\cdots j_k} \exp\big(-(n+k)P(\varphi) + \varphi_k(\omega_{j_1\cdots j_k}) + \varphi_n(\sigma^k(\omega_{j_1\cdots j_k}))\big) \\
&\leq L^2 \exp\big(-nP(\varphi) + \varphi_n(\omega)\big) \sum_{j_1\cdots j_k} \exp(-kP(\varphi))a_{j_1\cdots j_k} \\
&\leq L^2 \exp\big(-nP(\varphi) + \varphi_n(\omega)\big) \exp(-kP(\varphi))\alpha_k \\
&\leq L^3 \exp\big(-nP(\varphi) + \varphi_n(\omega)\big)
\end{aligned}$$

for every $k, n \in \mathbb{N}$ and $\omega \in C_{i_1\cdots i_n}$. Similarly, by (3.15), (3.20) and (3.22), we have

$$\nu(\sigma^{-k}C_{i_1\cdots i_n}) \geq L^{-2} \sum_{j_1\cdots j_k} \exp\big(-(n+k)P(\varphi) + \varphi_{n+k}(\omega_{j_1\cdots j_k})\big)$$

for every $k, n \in \mathbb{N}$ and $\omega_{j_1\cdots j_k} \in C_{j_1\cdots j_k i_1\cdots i_n}$. Taking $\omega \in C_{i_1\cdots i_n}$ as before, we thus obtain

$$\begin{aligned}
\nu(\sigma^{-k}C_{i_1\cdots i_n}) &\geq L^{-2} \exp\big(-nP(\varphi) + \varphi_n(\omega)\big) \sum_{j_1\cdots j_k} \exp(-kP(\varphi))e^{-D'}a_{j_1\cdots j_k} \\
&\geq L^{-3} \exp\big(-nP(\varphi) + \varphi_n(\omega)\big) \exp(-kP(\varphi))\alpha_k \\
&\geq L^{-3} \exp\big(-nP(\varphi) + \varphi_n(\omega)\big)
\end{aligned}$$

for every $k, n \in \mathbb{N}$ and $\omega \in C_{i_1\cdots i_n}$. Therefore,

$$\begin{aligned}
L^{-3} \exp\big(-nP(\varphi) + \varphi_n(\omega)\big) &\leq \frac{1}{m} \sum_{k=0}^{m-1} \nu(\sigma^{-k}C_{i_1\cdots i_n}) \\
&\leq L^3 \exp\big(-nP(\varphi) + \varphi_n(\omega)\big)
\end{aligned}$$

for every $m, n \in \mathbb{N}$ and $\omega \in C_{i_1\cdots i_n}$, and hence,

$$L^{-3} \leq \frac{\mu(C_{i_1\cdots i_n})}{\exp\big(-nP(\varphi) + \varphi_n(\omega)\big)} \leq L^3$$

for any sublimit μ of the sequence $(\mu_m)_{m\in\mathbb{N}}$. This shows that μ is a σ-invariant Gibbs measure for φ. □

Proofs of Theorem 3.4.4 are due to Sinai [185], Bowen [38], and Ruelle [165] (in chronological order). We note that under the hypotheses of Theorem 3.4.4 one can also show that there is a unique equilibrium measure for μ. It thus follows from Theorems 3.4.2 and 3.4.4 that there is a unique σ-invariant Gibbs measure μ for φ. Moreover, one can show that μ is ergodic. We refer to the proof of Theorem 10.1.9 for the corresponding arguments (although the proof is given in the more general setting of almost additive sequences, by considering the particular case of an additive sequence we readily obtain the corresponding proof in the additive case).

We also note that all the above notions and results have corresponding versions in the case of two-sided sequences, that is, for Gibbs measures in the space Σ_κ. We first recall the notion of Gibbs measure.

Definition 3.4.5. Given a continuous function $\varphi\colon \Sigma_\kappa \to \mathbb{R}$, we say that a probability measure μ in Σ_κ is a *Gibbs measure* for φ if there exists $K \ge 1$ such that

$$K^{-1} \le \frac{\mu(C_{i_{-n}\cdots i_n})}{\exp\left[-2nP(\varphi) + \sum_{l=-n}^{n} \varphi(\sigma^l(\omega))\right]} \le K$$

for every $(\cdots i_0 \cdots) \in \Sigma_\kappa$, $n \in \mathbb{N}$, and $\omega \in C_{i_{-n}\cdots i_n}$.

Writing

$$C'_{i_1\cdots i_m} = \bigcup_{i_{-m}\cdots i_0} C_{i_{-m}\cdots i_m},$$

one can easily verify that μ is a σ-invariant Gibbs measure for φ if and only if there exists $L \ge 1$ such that

$$L^{-1} \le \frac{\mu(C'_{i_1\cdots i_n})}{\exp\left[-nP(\varphi) + \sum_{l=0}^{n-1} \varphi(\sigma^l(\omega))\right]} \le L$$

for every $(\cdots i_0 \cdots) \in \Sigma_\kappa$, $n \in \mathbb{N}$, and $\omega \in C'_{i_1\cdots i_n}$. Repeating now arguments in the proof of Theorem 3.4.2 we obtain the following statement.

Theorem 3.4.6. *If a probability measure μ in Σ_κ is a σ-invariant Gibbs measure for φ, then it is also an equilibrium measure for φ.*

We have also a corresponding version of Theorem 3.4.4 for two-sided topological Markov chains.

Definition 3.4.7. Given a $\kappa \times \kappa$ matrix $A = (a_{ij})$ with $a_{ij} \in \{0, 1\}$ for each i and j, we consider the set

$$\Sigma_A = \left\{\omega \in \Sigma_\kappa : a_{i_n(\omega) i_{n+1}(\omega)} = 1 \text{ for every } n \in \mathbb{Z}\right\}.$$

Then the restriction $\sigma|\Sigma_A\colon \Sigma_A \to \Sigma_A$ is called the *(two-sided) topological Markov chain* or *subshift of finite type* with transition matrix A.

Repeating arguments in the proof of Theorem 3.4.4 we obtain the following statement.

Theorem 3.4.8. *Assume that A^q has only positive entries for some $q \in \mathbb{N}$. Then for the topological Markov chain $\sigma|\Sigma_A$, any Hölder continuous function $\varphi\colon \Sigma_A \to \mathbb{R}$ has at least one σ-invariant Gibbs measure.*

Part II

Nonadditive Thermodynamic Formalism

Chapter 4

Nonadditive Thermodynamic Formalism

This chapter is an introduction to the nonadditive thermodynamic formalism, which is an extension of the classical thermodynamic formalism where the topological pressure $P(\varphi)$ of a single function φ is replaced by the topological pressure $P(\Phi)$ of a sequence of functions $\Phi = (\varphi_n)_{n\in\mathbb{N}}$. The main motivation for this extension is a number of applications to a general class of invariant sets in the dimension theory of dynamical systems. In other words, in certain applications of dimension theory we are naturally led to consider sequences Φ that may satisfy no additivity between the functions φ_n. The nonadditive thermodynamic formalism also provides an appropriate unified framework for the development of the theory. Namely, the unique solution s of the so-called Bowen's equation $P(s\varphi) = 0$, where φ is a certain function associated to an invariant set, is often related to the Hausdorff dimension of the set. Moreover, virtually all known equations used to compute or to estimate the dimension of the invariant sets of a dynamical system are particular cases of this equation or of some generalization in the context of an appropriate extension of the classical thermodynamic formalism. See in particular Chapters 5 and 6, where we describe several applications of the nonadditive thermodynamic formalism respectively to repellers and to hyperbolic sets. After introducing the notion of nonadditive topological pressure as a Carathéodory dimension, we establish some of its basic properties. We also present nonadditive versions of the variational principle for the topological pressure and of Bowen's equation.

4.1 Nonadditive topological pressure

We introduce in this section the notion of nonadditive topological pressure for a continuous transformation of a compact metric space. The definition is given in terms of a Carathéodory dimension. This has in particular the advantages of

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including the case of noncompact sets, and of relating naturally to the notion of Hausdorff dimension, which is also defined in terms of a Carathéodory dimension.

Let $f\colon X \to X$ be a continuous transformation of a compact metric space (X, d). Let also $\mathcal{U}$ be a finite open cover of X. We shall use the same notation as in Section 2.2. In particular, for each $U \in \mathcal{W}_n(\mathcal{U})$, with $U = (U_1, \dots, U_n)$ for some $U_1, \dots, U_n \in \mathcal{U}$, we write $m(U) = n$, and we consider the open set $X(U)$ in (2.3). Again, a collection $\Gamma \subset \bigcup_{n\in\mathbb{N}} \mathcal{W}_n(\mathcal{U})$ is said to *cover* a set $Z \subset X$ if $\bigcup_{U\in\Gamma} X(U) \supset Z$.

Given a sequence $\Phi = (\varphi_n)_{n\in\mathbb{N}}$ of continuous functions $\varphi_n\colon X \to \mathbb{R}$, for each $n \in \mathbb{N}$ we define

$$\gamma_n(\Phi, \mathcal{U}) = \sup\big\{|\varphi_n(x) - \varphi_n(y)| : x, y \in X(U) \text{ for some } U \in \mathcal{W}_n(\mathcal{U})\big\}. \tag{4.1}$$

We always assume that the following property holds.

Definition 4.1.1. The sequence Φ is said to have *tempered variation* if

$$\limsup_{\operatorname{diam}\mathcal{U}\to 0} \limsup_{n\to\infty} \frac{\gamma_n(\Phi, \mathcal{U})}{n} = 0. \tag{4.2}$$

We note that since X is compact, it has finite open covers of arbitrarily small diameter. This ensures that we can indeed let $\operatorname{diam}\mathcal{U} \to 0$ in (4.2).

Now we follow the approach of Barreira in [5] to define the topological pressure of the sequence Φ as a Carathéodory dimension. The procedure is an elaboration of the construction in Section 2.2 due to Pesin and Pitskel' [153]. For each $n \in \mathbb{N}$ and $U \in \mathcal{W}_n(\mathcal{U})$, we write again

$$\varphi(U) = \begin{cases} \sup_{X(U)} \varphi_n & \text{if } X(U) \neq \varnothing, \\ -\infty & \text{if } X(U) = \varnothing. \end{cases}$$

Given $Z \subset X$ and $\alpha \in \mathbb{R}$, we define

$$M_Z(\alpha, \Phi, \mathcal{U}) = \lim_{n\to\infty} \inf_\Gamma \sum_{U\in\Gamma} \exp\big(-\alpha m(U) + \varphi(U)\big), \tag{4.3}$$

where the infimum is taken over all collections $\Gamma \subset \bigcup_{k\geq n} \mathcal{W}_k(\mathcal{U})$ covering Z. We also define

$$\underline{M}_Z(\alpha, \Phi, \mathcal{U}) = \liminf_{n\to\infty} \inf_\Gamma \sum_{U\in\Gamma} \exp\big(-\alpha m(U) + \varphi(U)\big), \tag{4.4}$$

$$\overline{M}_Z(\alpha, \Phi, \mathcal{U}) = \limsup_{n\to\infty} \inf_\Gamma \sum_{U\in\Gamma} \exp\big(-\alpha m(U) + \varphi(U)\big), \tag{4.5}$$

where the infimum is taken over all collections $\Gamma \subset \mathcal{W}_n(\mathcal{U})$ covering Z. One can show that when α goes from $-\infty$ to $+\infty$, each of the quantities in (4.3), (4.4), and (4.5) jumps from $+\infty$ to 0 at a unique value. Hence, we can define the numbers

$$\begin{aligned}
P_Z(\Phi, \mathcal{U}) &= \inf\bigl\{\alpha \in \mathbb{R} : M_Z(\alpha, \Phi, \mathcal{U}) = 0\bigr\},\\
\underline{P}_Z(\Phi, \mathcal{U}) &= \inf\bigl\{\alpha \in \mathbb{R} : \underline{M}_Z(\alpha, \Phi, \mathcal{U}) = 0\bigr\},\\
\overline{P}_Z(\Phi, \mathcal{U}) &= \inf\bigl\{\alpha \in \mathbb{R} : \overline{M}_Z(\alpha, \Phi, \mathcal{U}) = 0\bigr\}.
\end{aligned}$$

Theorem 4.1.2. *The limits*

$$\begin{aligned}
P_Z(\Phi) &= \lim_{\operatorname{diam}\mathcal{U}\to 0} P_Z(\Phi, \mathcal{U}),\\
\underline{P}_Z(\Phi) &= \lim_{\operatorname{diam}\mathcal{U}\to 0} \underline{P}_Z(\Phi, \mathcal{U}),\\
\overline{P}_Z(\Phi) &= \lim_{\operatorname{diam}\mathcal{U}\to 0} \overline{P}_Z(\Phi, \mathcal{U})
\end{aligned}$$

exist for each sequence Φ with tempered variation.

Proof. The proof is a slight modification of the proof of Theorem 2.2.1. Let $\mathcal{V}$ be a finite open cover of X with diameter smaller than the Lebesgue number of $\mathcal{U}$. As in the proof of Theorem 2.2.1, if $\Gamma \subset \bigcup_{k\in\mathbb{N}} \mathcal{W}_k(\mathcal{V})$ covers a set Z, then the collection $\{U(V) : V \in \Gamma\}$ in (2.5) also covers Z. Now let

$$\gamma(\mathcal{U}) = \limsup_{n\to\infty} \frac{\gamma_n(\Phi, \mathcal{U})}{n}.$$

Given $\varepsilon > 0$, we have $\gamma_n(\Phi, \mathcal{U})/n \le \gamma(\mathcal{U}) + \varepsilon$ for all sufficiently large n. We thus obtain

$$\varphi(U(V)) \le \varphi(V) + n(\gamma(\mathcal{U}) + \varepsilon)$$

for each $V \in \mathcal{W}_n(\mathcal{V})$, and hence,

$$M_Z(\alpha, \Phi, \mathcal{U}) \le M_Z(\alpha - \gamma(\mathcal{U}) - \varepsilon, \Phi, \mathcal{V}).$$

Therefore, $P_Z(\Phi, \mathcal{U}) \le P_Z(\Phi, \mathcal{V}) + \gamma(\mathcal{U}) + \varepsilon$, and

$$P_Z(\Phi, \mathcal{U}) - \gamma(\mathcal{U}) - \varepsilon \le \liminf_{\operatorname{diam}\mathcal{V}\to 0} P_Z(\Phi, \mathcal{V}).$$

On the other hand, the tempered variation property in (4.2) implies that $\gamma(\mathcal{U}) \to 0$ when $\operatorname{diam}\mathcal{U} \to 0$. Since ε is arbitrary, we conclude that

$$\limsup_{\operatorname{diam}\mathcal{U}\to 0} P_Z(\Phi, \mathcal{U}) \le \liminf_{\operatorname{diam}\mathcal{V}\to 0} P_Z(\Phi, \mathcal{V}),$$

and $P_Z(\Phi)$ is well defined. The existence of the other two limits can be obtained in a similar manner. □

The limits in Theorem 4.1.2 are precisely the topological pressures.

Definition 4.1.3. The number $P_Z(\Phi)$ is called the *nonadditive topological pressure*, and the numbers $\underline{P}_Z(\Phi)$ and $\overline{P}_Z(\Phi)$ are called respectively the *nonadditive lower and upper capacity topological pressures* of the sequence of functions Φ on the set Z (with respect to f).

We shall drop the prefix nonadditive whenever there is no danger of confusion. We emphasize that the set Z need not be compact neither f-invariant.

We also present briefly an equivalent characterization of the topological pressures. For each $n \in \mathbb{N}$ and $U \in \mathcal{W}_n(\mathcal{U})$, we define

$$\bar{\varphi}(U) = \begin{cases} d_U & \text{if } X(U) \neq \varnothing, \\ -\infty & \text{if } X(U) = \varnothing, \end{cases}$$

for any given number

$$d_U \in \Big[\inf_{X(U)} \varphi_n, \sup_{X(U)} \varphi_n \Big].$$

We can now replace $\varphi(U)$ by $\bar{\varphi}(U)$ in each of the expressions (4.3), (4.4), and (4.5) to define new functions L_Z, $\underline{L}_Z$, and $\overline{L}_Z$. We also define the numbers

$$\begin{aligned} R_Z(\Phi, \mathcal{U}) &= \inf \big\{ \alpha \in \mathbb{R} : L_Z(\alpha, \Phi, \mathcal{U}) = 0 \big\}, \\ \underline{R}_Z(\Phi, \mathcal{U}) &= \inf \big\{ \alpha \in \mathbb{R} : \underline{L}_Z(\alpha, \Phi, \mathcal{U}) = 0 \big\}, \\ \overline{R}_Z(\Phi, \mathcal{U}) &= \inf \big\{ \alpha \in \mathbb{R} : \overline{L}_Z(\alpha, \Phi, \mathcal{U}) = 0 \big\}. \end{aligned}$$

Corollary 4.1.4. *We have*

$$\begin{aligned} P_Z(\Phi) &= \lim_{\operatorname{diam} \mathcal{U} \to 0} R_Z(\Phi, \mathcal{U}), \\ \underline{P}_Z(\Phi) &= \lim_{\operatorname{diam} \mathcal{U} \to 0} \underline{R}_Z(\Phi, \mathcal{U}), \\ \overline{P}_Z(\Phi) &= \lim_{\operatorname{diam} \mathcal{U} \to 0} \overline{R}_Z(\Phi, \mathcal{U}) \end{aligned}$$

for each sequence Φ with tempered variation.

Proof. Given $\varepsilon > 0$, for all sufficiently large n we have

$$|\varphi_n(x) - \varphi_n(y)| \leq n(\gamma(\mathcal{U}) + \varepsilon)$$

for every $U \in \mathcal{W}_n(\mathcal{U})$ and $x, y \in X(U)$. This implies that

$$\varphi(U) \leq \bar{\varphi}(U) + n(\gamma(\mathcal{U}) + \varepsilon) \leq \varphi(U) + n(\gamma(\mathcal{U}) + \varepsilon).$$

We thus obtain

$$M_Z(\alpha, \Phi, \mathcal{U}) \leq L_Z\big(\alpha - \gamma(\mathcal{U}) - \varepsilon, \Phi, \mathcal{U}\big) \leq M_Z\big(\alpha - \gamma(\mathcal{U}) - \varepsilon, \Phi, \mathcal{U}\big),$$

and hence,

$$P_Z(\Phi, \mathcal{U}) \leq R_Z(\Phi, \mathcal{U}) + \gamma(\mathcal{U}) + \varepsilon \leq P_Z(\Phi, \mathcal{U}) + \gamma(\mathcal{U}) + \varepsilon.$$

Since ε is arbitrary and $\gamma(\mathcal{U}) \to 0$ when $\operatorname{diam} \mathcal{U} \to 0$, it follows from Theorem 4.1.2 that

$$\lim_{\operatorname{diam} \mathcal{U} \to 0} R_Z(\Phi, \mathcal{U}) = P_Z(\Phi).$$

Similar arguments apply to establish the identities for $\underline{P}_Z$ and $\overline{P}_Z$. □

The following example considers the particular case of additive sequences.

Example 4.1.5. Given a continuous function $\varphi \colon X \to \mathbb{R}$, we consider the sequence of functions $\varphi_n = \sum_{k=0}^{n-1} \varphi \circ f^k$. We have

$$|\varphi_n(x) - \varphi_n(y)| \le \sum_{k=0}^{n-1} |\varphi(f^k(x)) - \varphi(f^k(y))|,$$

and hence,

$$\begin{aligned} \frac{\gamma_n(\Phi, \mathcal{U})}{n} &\le \frac{1}{n} \sum_{k=0}^{n-1} \sup \left\{ |\varphi(x) - \varphi(y)| : x, y \in f^k(X(U)) \text{ for some } U \in \mathcal{W}_n(\mathcal{U}) \right\} \\ &\le \sup \left\{ |\varphi(x) - \varphi(y)| : x, y \in U \text{ for some } U \in \mathcal{U} \right\}, \end{aligned}$$

since $f^k(X(U)) \subset U_{k+1}$ for $k = 0, \ldots, n-1$. By the uniform continuity of φ, the last supremum converges to 0 when $\operatorname{diam} \mathcal{U} \to 0$. This shows that the tempered variation property in (4.2) holds for the additive sequence $(\varphi_n)_{n \in \mathbb{N}}$.

In view of Example 4.1.5, for additive sequences we recover the notion of topological pressure introduced by Pesin and Pitskel' in [153], and the notions of lower and upper capacity topological pressures introduced by Pesin in [151]. For $Z = X$, we obtain an equivalent description of the notion of topological pressure for compact sets introduced by Ruelle in [164] in the case of expansive maps, and by Walters in [194] in the general case (see Theorem 4.2.6). The nonadditive thermodynamic formalism also contains as a particular case a new formulation of the subadditive thermodynamic formalism earlier introduced by Falconer in [56] (see the related discussion in Section 7.5).

Now we write $\Phi = 0$ if $\varphi_n = 0$ for every $n \in \mathbb{N}$. The number $h(f|Z) := P_Z(0)$ coincides with the notion of topological entropy for noncompact sets introduced by Pesin and Pitskel' in [153], and is equivalent to the notion earlier introduced by Bowen in [37] (see [153]). The numbers

$$\underline{h}(f|Z) := \underline{P}_Z(0) \quad \text{and} \quad \overline{h}(f|Z) := \overline{P}_Z(0)$$

coincide respectively with the notions of lower and upper capacity topological entropies introduced by Pesin in [151].

4.2 Properties of the pressure

We describe in this section some properties of the pressure functions. Namely, after describing how the pressures vary with the data used to define them, that is, the set Z and the sequence of functions Φ, we obtain several alternative formulas for the lower and upper capacity pressures. In particular, we obtain formulas in terms of separated sets. We also consider the particular case of subadditive sequences, which often occur in the applications, and we show that for compact invariant sets the three pressure functions coincide.

4.2.1 Dependence on the data

Following [5], we first describe how the pressure functions vary with the set Z and the sequence of functions Φ. We start with the dependence on the set.

Theorem 4.2.1. *The following properties hold:*

1.
$$P_Z(\Phi, \mathcal{U}) \le \underline{P}_Z(\Phi, \mathcal{U}) \le \overline{P}_Z(\Phi, \mathcal{U}), \tag{4.6}$$
and hence,
$$P_Z(\Phi) \le \underline{P}_Z(\Phi) \le \overline{P}_Z(\Phi); \tag{4.7}$$

2. *if* $Z_1 \subset Z_2$, *then*
$$P_{Z_1}(\Phi) \le P_{Z_2}(\Phi), \quad \underline{P}_{Z_1}(\Phi) \le \underline{P}_{Z_2}(\Phi), \quad \overline{P}_{Z_1}(\Phi) \le \overline{P}_{Z_2}(\Phi); \tag{4.8}$$

3. *if* $Z = \bigcup_{i \in I} Z_i$ *is a union of sets* $Z_i \subset X$, *with* I *at most countable, then*
$$P_Z(\Phi) = \sup_{i \in I} P_{Z_i}(\Phi).$$

Proof. For the first property, we note that since $\mathcal{W}_n(\mathcal{U}) \subset \bigcup_{k \ge n} \mathcal{W}_k(\mathcal{U})$, it follows readily from the definitions that

$$M_Z(\alpha, \Phi, \mathcal{U}) \le \underline{M}_Z(\alpha, \Phi, \mathcal{U}) \le \overline{M}_Z(\alpha, \Phi, \mathcal{U}).$$

Therefore, (4.6) holds, and taking limits when $\operatorname{diam} \mathcal{U} \to 0$ yields (4.7).

The second property also follows easily from the definitions, since

$$\begin{aligned} M_{Z_1}(\alpha, \Phi, \mathcal{U}) &\le M_{Z_2}(\alpha, \Phi, \mathcal{U}), \\ \underline{M}_{Z_1}(\alpha, \Phi, \mathcal{U}) &\le \underline{M}_{Z_2}(\alpha, \Phi, \mathcal{U}), \\ \overline{M}_{Z_1}(\alpha, \Phi, \mathcal{U}) &\le \overline{M}_{Z_2}(\alpha, \Phi, \mathcal{U}). \end{aligned}$$

Namely, the first inequality implies that $P_{Z_1}(\Phi, \mathcal{U}) \le P_{Z_2}(\Phi, \mathcal{U})$, and there are corresponding inequalities for the capacity topological pressures. Taking limits when $\operatorname{diam} \mathcal{U} \to 0$ we obtain the inequalities in (4.8).

Now we establish the third property. Clearly,

$$P_Z(\Phi) = \lim_{\operatorname{diam}\mathcal{U}\to 0} P_Z(\Phi,\mathcal{U}) \geq \lim_{\operatorname{diam}\mathcal{U}\to 0} \sup_{i\in I} P_{Z_i}(\Phi,\mathcal{U}), \tag{4.9}$$

in view of property 2. For the reverse inequality, take $\alpha > \sup_{i\in I} P_{Z_i}(\Phi,\mathcal{U})$. Then $M_{Z_i}(\alpha,\Phi,\mathcal{U}) = 0$ for each $i \in I$. This implies that given $\delta > 0$ and $n \in \mathbb{N}$, for each $i \in I$ there is a collection $\Gamma_i \subset \bigcup_{k\geq n} \mathcal{W}_k(\mathcal{U})$ covering Z_i such that

$$\sum_{U\in\Gamma_i} \exp\big(-\alpha m(U) + \varphi(U)\big) < \frac{\delta}{2^i}.$$

Then the collection $\Gamma = \bigcup_{i\in I} \Gamma_i \subset \bigcup_{k\geq n} \mathcal{W}_k(\mathcal{U})$ covers the union $Z = \bigcup_{i\in I} Z_i$, and we have

$$\sum_{U\in\Gamma} \exp\big(-\alpha m(U) + \varphi(U)\big) \leq \sum_{i\in I} \frac{\delta}{2^i} \leq \delta.$$

Letting $n \to \infty$ we obtain $M_Z(\alpha,\Phi,\mathcal{U}) \leq \delta$, and hence $M_Z(\alpha,\Phi,\mathcal{U}) = 0$, since δ is arbitrary. Therefore, $\alpha \geq P_Z(\Phi,\mathcal{U})$, and letting $\alpha \to \sup_{i\in I} P_{Z_i}(\Phi,\mathcal{U})$ yields

$$\lim_{\operatorname{diam}\mathcal{U}\to 0} \sup_{i\in I} P_{Z_i}(\Phi,\mathcal{U}) \geq P_Z(\Phi). \tag{4.10}$$

To complete the proof we show that the limit and the supremum in (4.10) can be interchanged. Since

$$\sup_{i\in I} P_{Z_i}(\Phi,\mathcal{U}) \geq P_{Z_j}(\Phi,\mathcal{U})$$

for each $j \in I$, we have

$$\beta := \lim_{\operatorname{diam}\mathcal{U}\to 0} \sup_{i\in I} P_{Z_i}(\Phi,\mathcal{U}) \geq \sup_{j\in I} \lim_{\operatorname{diam}\mathcal{U}\to 0} P_{Z_j}(\Phi,\mathcal{U}). \tag{4.11}$$

Moreover, given $\delta > 0$ we have

$$\sup_{i\in I} P_{Z_i}(\Phi,\mathcal{U}) < \beta + \delta$$

for any finite open cover $\mathcal{U}$ of X with sufficiently small diameter. Therefore, $P_{Z_i}(\Phi,\mathcal{U}) < \beta + \delta$, and

$$P_{Z_i}(\Phi) \leq \beta + \delta \quad \text{for each} \quad i \in I.$$

This implies that

$$\sup_{i\in I} \lim_{\operatorname{diam}\mathcal{U}\to 0} P_{Z_i}(\Phi,\mathcal{U}) = \sup_{i\in I} P_{Z_i}(\Phi) \leq \beta + \delta.$$

Letting $\delta \to 0$ we obtain

$$\sup_{i\in I} \lim_{\operatorname{diam}\mathcal{U}\to 0} P_{Z_i}(\Phi,\mathcal{U}) \leq \lim_{\operatorname{diam}\mathcal{U}\to 0} \sup_{i\in I} P_{Z_i}(\Phi,\mathcal{U}),$$

which together with (4.9), (4.10), and (4.11) yields the third property in the theorem. □

Now we define a semi-norm in the space of sequences Φ in X by

$$\|\Phi\| = \limsup_{n\to\infty} \frac{1}{n} \sup\bigl\{|\varphi_n(x)| : x \in X\bigr\}.$$

The following result shows that the pressures are continuous with respect to this semi-norm.

Theorem 4.2.2. *For any sequences $\Phi = (\varphi_n)_{n\in\mathbb{N}}$ and $\Psi = (\psi_n)_{n\in\mathbb{N}}$ with tempered variation the following properties hold:*

1. *if $\varphi_n \le \psi_n$ for all sufficiently large n, then*

$$P_Z(\Phi,\mathcal{U}) \le P_Z(\Psi,\mathcal{U}),\ \underline{P}_Z(\Phi,\mathcal{U}) \le \underline{P}_Z(\Psi,\mathcal{U}),\ \overline{P}_Z(\Phi,\mathcal{U}) \le \overline{P}_Z(\Psi,\mathcal{U});$$

2. *if the pressure functions attain only finite values, then*

$$\begin{aligned} |P_Z(\Phi,\mathcal{U}) - P_Z(\Psi,\mathcal{U})| &\le \|\Phi-\Psi\|,\\ |\underline{P}_Z(\Phi,\mathcal{U}) - \underline{P}_Z(\Psi,\mathcal{U})| &\le \|\Phi-\Psi\|,\\ |\overline{P}_Z(\Phi,\mathcal{U}) - \overline{P}_Z(\Psi,\mathcal{U})| &\le \|\Phi-\Psi\|. \end{aligned} \tag{4.12}$$

Proof. The first property follows immediately from the definitions. Now we establish the second property. Given $\varepsilon > 0$, we have

$$|\varphi_n - \psi_n| \le n(\|\Phi-\Psi\| + \varepsilon)$$

for all sufficiently large n. Hence,

$$M_Z\bigl(\alpha + \|\Phi-\Psi\| + \varepsilon, \Psi, \mathcal{U}\bigr) \le M_Z(\alpha,\Phi,\mathcal{U}) \le M_Z\bigl(\alpha - \|\Phi-\Psi\| - \varepsilon, \Psi, \mathcal{U}\bigr),$$

and

$$P_Z(\Psi,\mathcal{U}) - \|\Phi-\Psi\| - \varepsilon \le P_Z(\Phi,\mathcal{U}) + \|\Phi-\Psi\| + \varepsilon.$$

Since ε is arbitrary, this yields the first inequality in (4.12). Similar arguments establish the other inequalities. □

4.2.2 Characterizations of the capacity pressures

We establish in this section several alternative formulas for the lower and upper capacity topological pressures, following as much as possible the related approach in Section 2.2 for the classical topological pressure. In particular, we obtain formulas in terms of separated sets. We emphasize that in strong contrast to what happens in Section 2.2, here the set Z need not be compact neither invariant.

Given a finite open cover $\mathcal{U}$ of X and a set $Z \subset X$, for each $n \in \mathbb{N}$ we define

$$\mathcal{Z}_n(Z,\Phi,\mathcal{U}) = \inf_{\Gamma} \sum_{U\in\Gamma} \exp \sup_{X(U)} \varphi_n, \tag{4.13}$$

where the infimum is taken over all collections $\Gamma \subset \mathcal{W}_n(\mathcal{U})$ covering Z.

Proposition 4.2.3. *We have*

$$\underline{P}_Z(\Phi, \mathcal{U}) = \liminf_{n\to\infty} \frac{1}{n} \log \mathcal{Z}_n(Z, \Phi, \mathcal{U}) \tag{4.14}$$

and

$$\overline{P}_Z(\Phi, \mathcal{U}) = \limsup_{n\to\infty} \frac{1}{n} \log \mathcal{Z}_n(Z, \Phi, \mathcal{U}). \tag{4.15}$$

Proof. We note that

$$\underline{M}_Z(\alpha, \Phi, \mathcal{U}) = \liminf_{n\to\infty} \left(e^{-\alpha n} \mathcal{Z}_n(Z, \Phi, \mathcal{U})\right)$$

and

$$\overline{M}_Z(\alpha, \Phi, \mathcal{U}) = \limsup_{n\to\infty} \left(e^{-\alpha n} \mathcal{Z}_n(Z, \Phi, \mathcal{U})\right).$$

Let us take $\alpha > \underline{P}_Z(\Phi, \mathcal{U})$. Then there exists a sequence $m_n \nearrow +\infty$ of positive integers such that

$$e^{-\alpha m_n} \mathcal{Z}_{m_n}(Z, \Phi, \mathcal{U}) < 1 \quad \text{for each} \quad n \in \mathbb{N}.$$

Therefore,

$$\frac{1}{m_n} \log \mathcal{Z}_{m_n}(Z, \Phi, \mathcal{U}) < \alpha,$$

and hence,

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathcal{Z}_n(Z, \Phi, \mathcal{U}) \le \underline{P}_Z(\Phi, \mathcal{U}). \tag{4.16}$$

Now we take $\alpha < \underline{P}_Z(\Phi, \mathcal{U})$. Then $e^{-\alpha n} \mathcal{Z}_n(Z, \Phi, \mathcal{U}) > 1$ for any sufficiently large n. This implies that

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathcal{Z}_n(Z, \Phi, \mathcal{U}) \ge \alpha,$$

and hence,

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathcal{Z}_n(Z, \Phi, \mathcal{U}) \ge \underline{P}_Z(\Phi, \mathcal{U}).$$

Together with (4.16) this establishes identity (4.14). The proof of identity (4.15) is entirely analogous. □

Now we obtain formulas for the lower and upper capacity topological pressures in terms of separated sets and in terms of covers by d_n-balls, in a similar manner to that in Section 2.2 for the classical topological pressure. We follow closely the appendix of [5].

Given a set $Z \subset X$ and $\varepsilon > 0$, for each $n \in \mathbb{N}$ we define

$$R_n(Z, \Phi, \varepsilon) = \sup_E \sum_{x \in E \cap Z} \exp \varphi_n(x),$$

where the supremum is taken over all (n,ε)-separated sets $E \subset X$, and also

$$S_n(Z,\Phi,\varepsilon) = \inf_{\mathcal{V}} \sum_{V\in\mathcal{V}} \exp b_V,$$

where the infimum is taken over all finite open covers $\mathcal{V}$ of Z by d_n-balls of radius ε, and where b_V is any given number in the interval $[\inf_V \varphi_n, \sup_V \varphi_n]$.

Theorem 4.2.4 ([5]). *We have*

$$\begin{aligned}\underline{P}_Z(\Phi) &= \lim_{\varepsilon\to 0} \liminf_{n\to\infty} \frac{1}{n}\log R_n(Z,\Phi,\varepsilon)\\ &= \lim_{\varepsilon\to 0} \liminf_{n\to\infty} \frac{1}{n}\log S_n(Z,\Phi,\varepsilon)\end{aligned} \tag{4.17}$$

and

$$\begin{aligned}\overline{P}_Z(\Phi) &= \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n}\log R_n(Z,\Phi,\varepsilon)\\ &= \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n}\log S_n(Z,\Phi,\varepsilon).\end{aligned} \tag{4.18}$$

Proof. Given $\varepsilon > 0$, let $\mathcal{U}$ be a finite open cover of X with diameter less than ε. We note that distinct elements of an (n,ε)-separated set E are in distinct elements of the open cover

$$\mathcal{U}_n = \big\{X(U) : U \in \mathcal{W}_n(\mathcal{U}), X(U)\cap Z \neq \varnothing\big\}.$$

This implies that

$$R_n(Z,\Phi,\varepsilon) \le \mathcal{Z}_n(Z,\Phi,\mathcal{U}). \tag{4.19}$$

Now let 2δ be a Lebesgue number of the cover $\mathcal{U}$, and take $x \in X$. For $k = 0,\dots,n-1$, let us take $U_{k+1} \in \mathcal{U}$ such that $B(f^k(x),\delta) \subset U_{k+1}$. Then the d_n-ball of radius δ centered at x satisfies

$$B_n(x,\delta) = \bigcap_{k=0}^{n-1} f^{-k} B(f^k(x),\delta) \subset X(U),$$

where $U = (U_1,\dots,U_n)$. Therefore, for each finite open cover $\mathcal{V}$ of Z by d_n-balls of radius δ intersecting Z, we have

$$b_V \le \inf_V \varphi_n + \gamma_n(\Phi,\mathcal{U}) \tag{4.20}$$

for any $V \in \mathcal{V}$. Furthermore, each element of any such cover $\mathcal{V}$ is contained in an element of $\mathcal{U}_n$. Therefore,

$$\inf_{\Gamma} \sum_{U\in\Gamma} \exp \inf_{X(U)} \varphi_n \le S_n(Z,\Phi,\delta), \tag{4.21}$$

where the infimum is taken over all collections $\Gamma \subset \mathcal{W}_n(\mathcal{U})$ covering Z.

On the other hand, since the d_n-balls of radius δ centered at the points of some (n,δ)-separated set E may not cover Z, we have

$$\inf_{\mathcal{V}} \sum_{V \in \mathcal{V}} \exp \inf_{V} \varphi_n \le R_n(Z,\Phi,\delta), \tag{4.22}$$

where the infimum is taken over all finite open covers $\mathcal{V}$ of Z by d_n-balls of radius δ.

Applying successively the inequalities (4.19), (4.20), (4.21), and (4.22), we obtain

$$\begin{aligned} R_n(Z,\Phi,\varepsilon) &\le \mathcal{Z}_n(Z,\Phi,\mathcal{U}) \\ &\le \inf_{\Gamma} \sum_{U \in \Gamma} \exp \inf_{X(U)} \varphi_n \times e^{\gamma_n(\Phi,\mathcal{U})} \\ &\le S_n(Z,\Phi,\delta) \times e^{\gamma_n(\Phi,\mathcal{U})} \\ &\le \inf_{\mathcal{V}} \sum_{V \in \mathcal{V}} \exp \inf_{V} \varphi_n \times e^{2\gamma_n(\Phi,\mathcal{U})} \\ &\le R_n(Z,\Phi,\delta) \times e^{2\gamma_n(\Phi,\mathcal{U})}. \end{aligned} \tag{4.23}$$

Since $\operatorname{diam} \mathcal{U} \to 0$ and $\delta \to 0$ when $\varepsilon \to 0$, the desired formulas in (4.17) and (4.18) follow now readily from (4.23) together with (4.14) and (4.15). □

4.2.3 Subadditive sequences

We give in this section alternative characterizations of the nonadditive topological pressures in the particular case of subadditive sequences. We note that these often occur in the applications (see in particular Chapters 5 and 8). As a consequence, we show that for compact invariant sets the three pressure functions coincide. We refer to Chapter 7 for a more complete development of the nonadditive thermodynamic formalism for subadditive sequences, including a variational principle for the topological pressure and the construction of equilibrium measures.

We first recall the notion of subadditive sequence.

Definition 4.2.5. A sequence of functions Φ is said to be *subadditive* if

$$\varphi_{m+n}(x) \le \varphi_n(x) + \varphi_m(f^n(x))$$

for every $m,n \in \mathbb{N}$ and $x \in X$.

The following is a characterization of the nonadditive topological pressure for subadditive sequences.

Theorem 4.2.6. *If $Z \subset X$ is f-invariant and Φ is subadditive, then:*

1.

$$\begin{aligned} \hat{P}_Z(\Phi,\mathcal{U}) := \underline{P}_Z(\Phi,\mathcal{U}) &= \overline{P}_Z(\Phi,\mathcal{U}) \\ &= \lim_{n\to\infty} \frac{1}{n} \log \mathcal{Z}_n(Z,\Phi,\mathcal{U}); \end{aligned}$$

2. *if in addition Z is compact, then* $P_Z(\Phi, \mathcal{U}) = \hat{P}_Z(\Phi, \mathcal{U})$.

Proof. The proof of the first property is an elaboration of the proof of Lemma 2.2.4. Given $\Gamma_m \subset \mathcal{W}_m(\mathcal{U})$ and $\Gamma_n \subset \mathcal{W}_n(\mathcal{U})$, we define

$$\Gamma_{m,n} = \big\{UV : U \in \Gamma_m, V \in \Gamma_n\big\} \subset \mathcal{W}_{n+m}(\mathcal{U}).$$

Since Z is f-invariant, if Γ_m and Γ_n cover Z, then the collection $\Gamma_{m,n}$ also covers Z. Since Φ is subadditive, we have

$$\varphi(UV) \le \varphi(U) + \varphi(V) \quad \text{for each} \quad UV \in \Gamma_{m,n}.$$

Therefore,

$$\begin{aligned}\mathcal{Z}_{m+n}(Z, \Phi, \mathcal{U}) &\le \sum_{UV \in \Gamma_{m,n}} \exp \varphi(UV)\\ &\le \sum_{U \in \Gamma_m} \exp \varphi(U) \times \sum_{V \in \Gamma_n} \exp \varphi(V),\end{aligned}$$

and

$$\mathcal{Z}_{m+n}(Z, \Phi, \mathcal{U}) \le \mathcal{Z}_m(Z, \Phi, \mathcal{U})\mathcal{Z}_n(Z, \Phi, \mathcal{U}).$$

This implies that the sequence $\frac{1}{n} \log \mathcal{Z}_n(Z, \Phi, \mathcal{U})$ converges. The first property follows now readily from (4.14) and (4.15).

To establish the second property, we proceed in a similar manner to that in the proof of Lemma 2.2.5. Namely, let $\Gamma \subset \bigcup_{n \in \mathbb{N}} \mathcal{W}_n(\mathcal{U})$ be a collection covering Z. Since Z is compact, we may assume that Γ is finite, and thus that there exists an integer $q \in \mathbb{N}$ such that $\Gamma \subset \bigcup_{n \le q} \mathcal{W}_n(\mathcal{U})$. Let again

$$\Gamma^n = \big\{(U_1, \dots, U_n) : U_i \in \Gamma\big\}$$

for each $n \in \mathbb{N}$. Since Z is f-invariant, the collection Γ^n covers Z for each $n \in \mathbb{N}$. Since Φ is subadditive, if $U = (U_1, \dots, U_n) \in \Gamma^n$ then $\varphi(U) \le \sum_{i=1}^n \varphi(U_i)$. Thus, given $\alpha \in \mathbb{R}$ and using the notation $N(\Gamma)$ in (2.11), again we obtain inequality (2.12). Given $\alpha > P_Z(\Phi, \mathcal{U})$, there exist $m \in \mathbb{N}$ and a collection $\Gamma \subset \bigcup_{n \ge m} \mathcal{W}_n(\mathcal{U})$ covering Z such that $N(\Gamma) < 1$. For the collection Γ^∞ in (2.13), it thus follows from (2.12) that $N(\Gamma^\infty) < \infty$.

The remaining part of the argument is taken from [42]. Given $n \ge 2q$, let $\Lambda_n \subset \Gamma^\infty$ be the collection of all vectors $U = (U_1, \dots, U_k)$ such that $U_i \in \Gamma$ for $i = 1, \dots, k$, and

$$\sum_{i=1}^{k-1} m(U_i) < n - q \le \sum_{i=1}^{k} m(U_i).$$

Clearly, $n - q \le m(U) < n$ for each $U \in \Lambda_n$. Moreover, since Z is f-invariant, the collection Λ_n covers Z. We also consider the collection

$$\Lambda_n' = \big\{UV : U \in \Lambda_n, V \in \mathcal{W}_{n-m(U)}(\mathcal{U})\big\} \subset \mathcal{W}_n(\mathcal{U}).$$

Then

$$\begin{aligned} N(\Lambda_n') &\le \sum_{U\in\Lambda_n, V\in\mathcal{W}_{n-m(U)}(\mathcal{U})} \exp\big(-\alpha n+\varphi(UV)\big) \\ &\le \sum_{U\in\Lambda_n, V\in\mathcal{W}_{n-m(U)}(\mathcal{U})} \exp\big[-\alpha(m(U)+m(V))+\varphi(U)+\varphi(V)\big] \\ &\le \sum_{U\in\Lambda_n} \exp\big(-\alpha m(U)+\varphi(U)\big)\times\sum_{V\in\mathcal{V}}\exp\big(-\alpha m(V)+\varphi(V)\big) \\ &\le N(\Gamma^\infty)\sum_{V\in\mathcal{V}}\exp\big(-\alpha m(V)+\varphi(V)\big)<\infty, \end{aligned}$$

where we have set $\mathcal{V}=\bigcup_{j=1}^q \mathcal{W}_j(\mathcal{U})$. Since the collection Λ_n' covers Z, we obtain $\overline{M}_Z(\alpha,\Phi,\mathcal{U})<\infty$ and $\alpha>\overline{P}_Z(\Phi,\mathcal{U})$. Therefore, $P_Z(\Phi,\mathcal{U})\ge\overline{P}_Z(\Phi,\mathcal{U})$, and the first property in Theorem 4.2.1 implies the desired result. □

Theorem 4.2.6 was established by Barreira in [5], assuming in the second property that there exists $K>0$ such that $\varphi_n\le\varphi_{n+1}+K$ for all $n\in\mathbb{N}$. This assumption was removed by Cao, Feng and Huang in [42] when $Z=X$, although their argument readily extends to arbitrary sets.

The following statement is a simple consequence of Theorems 4.2.4 and 4.2.6.

Theorem 4.2.7. *If $Z\subset X$ is f-invariant and Φ is subadditive, then*

$$\begin{aligned} \hat{P}_Z(\Phi) &= \lim_{\operatorname{diam}\mathcal{U}\to 0}\lim_{n\to\infty}\frac{1}{n}\log\mathcal{Z}_n(Z,\Phi,\mathcal{U}) \\ &= \lim_{\varepsilon\to 0}\liminf_{n\to\infty}\frac{1}{n}\log R_n(Z,\Phi,\varepsilon) \\ &= \lim_{\varepsilon\to 0}\liminf_{n\to\infty}\frac{1}{n}\log S_n(Z,\Phi,\varepsilon) \\ &= \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log R_n(Z,\Phi,\varepsilon) \\ &= \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log S_n(Z,\Phi,\varepsilon). \end{aligned}$$

4.3 Variational principle

We establish in this section a nonadditive version of the variational principle for the topological pressure. We follow the approach of Barreira in [5] that elaborates on work of Pesin and Pitskel' in [153] in the additive case. Again, the set Z need not be compact neither invariant, which substantially complicates the proof. We also follow as much as possible some classical arguments of Bowen [39].

We continue to denote by $\mathcal{M}_f$ the set of all f-invariant probability measures in X. For each f-invariant measurable set $Z\subset X$, we denote by $\mathcal{M}_f(Z)$ the set of all measures $\mu \in \mathcal{M}_f$ such that $\mu(Z) = 1$. For each $\mu \in \mathcal{M}_f(Z)$, we denote by $h_\mu(f|Z)$ the entropy of $f|Z$ with respect to μ (see Section 1.4.2 for the definition).

Given $x \in X$ and $n \in \mathbb{N}$, we define a probability measure in X by

$$\mu_{x,n} = \frac{1}{n}\sum_{k=0}^{n-1} \delta_{f^k(x)},$$

where δ_y is the delta-measure at y. Let also $V(x)$ be the set of all sublimits of the sequence $(\mu_{x,n})_{n\in\mathbb{N}}$. It is easy to see that $\varnothing \neq V(x) \subset \mathcal{M}_f$ for every $x \in X$. Moreover, for each $\mu \in \mathcal{M}_f$ we consider the set

$$\mathcal{L}(Z) = \bigl\{x \in Z : V(x) \cap \mathcal{M}_f(Z) \neq \varnothing\bigr\}.$$

One can easily verify that $\mathcal{L}(Z)$ is f-invariant and measurable.

Theorem 4.3.1 ([5]). *Let Φ be a sequence of continuous functions with tempered variation, and let $Z \subset X$ be an f-invariant measurable set. If there exists a continuous function $\psi\colon X \to \mathbb{R}$ such that*

$$\varphi_{n+1} - \varphi_n \circ f \to \psi \text{ uniformly on } Z \text{ when } n \to \infty, \tag{4.24}$$

then

$$P_{\mathcal{L}(Z)}(\Phi) = \sup_{\mu\in\mathcal{M}_f(Z)} \left(h_\mu(f|Z) + \int_Z \psi \, d\mu\right). \tag{4.25}$$

Proof. Take $x \in \mathcal{L}(Z)$. For each measure $\mu \in V(x) \cap \mathcal{M}_f(Z)$, given $\delta > 0$ there exists an increasing sequence $(n_j)_{j\in\mathbb{N}} \subset \mathbb{N}$ such that

$$\left|\frac{1}{n_j}\sum_{k=0}^{n_j-1} \psi(f^k(x)) - \int_Z \psi \, d\mu\right| < \delta$$

for all $j \in \mathbb{N}$. Now we write

$$\psi_k = \varphi_{k+1} - \varphi_k \circ f - \psi. \tag{4.26}$$

Since

$$\varphi_n - \sum_{k=0}^{n-1} \psi \circ f^k = \sum_{k=1}^{n-1} \psi_k \circ f^{n-k-1} + \varphi_1 \circ f^{n-1}$$

and Z is f-invariant, we obtain

$$\begin{aligned}\left|\frac{\varphi_{n_j}(x)}{n_j} - \int_Z \psi \, d\mu\right| &\le \left|\frac{1}{n_j}\left(\varphi_{n_j}(x) - \sum_{k=0}^{n_j-1} \psi(f^k(x))\right)\right| + \delta \\ &\le \frac{1}{n_j}\left(\sum_{k=1}^{n_j-1} \|\psi_k\|_\infty + \|\varphi_1\|_\infty\right) + \delta,\end{aligned}$$

where $\|g\|_\infty = \sup_{x\in Z}|g(x)|$. By (4.24), there exists $M \in \mathbb{N}$ such that $\|\psi_n\|_\infty < \delta$ for every $n \geq M$. Therefore,

$$\left|\frac{\varphi_{n_j}(x)}{n_j} - \int_Z \psi\, d\mu\right| \leq \frac{1}{n_j}\left((n_j - M)\delta + \sum_{k=1}^{M-1}\|\psi_k\|_\infty + \|\varphi_1\|_\infty\right) + \delta \leq 2\delta, \quad (4.27)$$

provided that j is sufficiently large.

Now let E be a finite set. Given $k \in \mathbb{N}$ and $a = (a_1, \ldots, a_k) \in E^k$, we define a probability measure μ_a in E by

$$\mu_a(e) = \frac{1}{k}\operatorname{card}\{j : a_j = e\} \quad (4.28)$$

for each $e \in E$. Let also

$$H(a) = -\sum_{e\in E}\mu_a(e)\log\mu_a(e). \quad (4.29)$$

We modify the proof of Lemma 2.15 in [39] to obtain the following.

Lemma 4.3.2. *Take $x \in \mathcal{L}(Z)$, $\mu \in V(x) \cap \mathcal{M}_f(Z)$, and $\delta > 0$. Given a finite open cover $\mathcal{U}$ of X, there exist $m, N \in \mathbb{N}$ with N arbitrarily large and $U \in \mathcal{W}_N(\mathcal{U})$ such that:*

1. *$x \in X(U)$ and*
$$\varphi_N(x) \leq N\left(\int_Z \psi\, d\mu + 2\delta\right);$$
2. *U contains a subvector V of length $km \geq N - m$ that seen as an element of $(\mathcal{U}^m)^k$ satisfies*
$$H(V) \leq m(h_\mu(f|Z) + \delta).$$

Proof of the lemma. Write $\mathcal{U} = \{U_1, \ldots, U_r\}$ and let $\xi = \{C_1, \ldots, C_r\}$ be a measurable partition of X such that $\overline{C_i} \subset U_i$ for each i. Let us also take $\varepsilon > 0$ and $m \in \mathbb{N}$ such that

$$\frac{1}{m}H_\mu\left(\bigvee_{j=0}^{m-1} f^{-j}\xi\right) < h_\mu(f,\xi) + \frac{\varepsilon}{2} \leq h_\mu(f) + \frac{\varepsilon}{2}.$$

Finally, let $(n_j)_{j\in\mathbb{N}} \subset \mathbb{N}$ be a sequence such that $\mu_{x,n_j} \to \mu$ when $j \to \infty$. If m_j is the least multiple of m such that $m_j \geq n_j$, then

$$\mu_{x,m_j} = \frac{n_j}{m_j}\mu_{x,n_j} + \frac{m_j - n_j}{m_j}\mu_{f^{n_j}(x),m_j-n_j} \to \mu$$

when $j \to \infty$. Hence, without loss of generality, we can always assume that $n_j = mk_j$ for some integer k_j.

Now let $D_1, \dots, D_t$ be the nonempty elements of the partition $\bigvee_{j=0}^{m-1} f^{-j}\xi$. Given $\beta > 0$, for each $i = 1, \dots, t$, let $K_i \subset D_i$ be a compact set such that $\mu(D_i \setminus K_i) < \beta$. Each set D_i is contained in some element of $\bigvee_{j=0}^{m-1} f^{-j}\mathfrak{U}$, and there exists an open set $V_i \supset K_i$ with the same property. We may also assume that $V_i \cap V_j = \varnothing$ whenever $i \neq j$. Moreover, we consider a measurable partition $\{V_1', \dots, V_t'\}$ of X such that V_i' is contained in some element of $\bigvee_{j=0}^{m-1} f^{-j}\mathfrak{U}$ and $V_i' \supset V_i$ for each i.

Given $j \in \mathbb{N}$, let M_i be the number of integers $p \in [0, n_j) \cap \mathbb{Z}$ such that $f^p(x) \in V_i'$, and set $M_{i,r} = M_i \pmod m$. We define $p_{i,r} = M_{i,r}/k_j$ and

$$p_i = \frac{M_i}{n_j} = \frac{1}{m}(p_{i,0} + \cdots + p_{i,m-1}).$$

Since $\mu_{x,n_j} \to \mu$, when $j \to \infty$, we have

$$\liminf_{j\to\infty} p_i \geq \mu(K_i) \geq \mu(D_i) - \beta,$$

and

$$\limsup_{j\to\infty} p_i \leq \mu(K_i) + t\beta \leq \mu(D_i) + t\beta.$$

Provided that β is sufficiently small and j is sufficiently large, we thus obtain

$$-\frac{1}{m}\sum_{i=1}^t p_i \log p_i \leq -\frac{1}{m}\sum_{i=1}^t \mu(D_i)\log\mu(D_i) + \frac{\varepsilon}{2} \leq h_\mu(f) + \varepsilon. \tag{4.30}$$

By the convexity of the function χ in (2.33), we have

$$\chi(p_i) \geq \sum_{r=0}^{m-1} \frac{1}{m}\chi(p_{i,r}),$$

and thus,

$$\sum_{i=1}^t \chi(p_i) \geq \frac{1}{m}\sum_{r=0}^{m-1}\sum_{i=1}^t \chi(p_{i,r}).$$

This implies that $\sum_{i=1}^t \chi(p_{i,r}) \leq \sum_{i=1}^t \chi(p_i)$ for some $r \in [0, m)$, and by (4.30) we obtain

$$\frac{1}{m}\sum_{i=1}^t \chi(p_{i,r}) \leq h_\mu(f) + \varepsilon. \tag{4.31}$$

For $N = n_j + r$, we construct a vector $U = (U_0, \dots, U_{N-1}) \in \mathfrak{U}^N$ as follows. For $s < r$, take $U_s \in \mathfrak{U}$ containing $f^s(x)$. For each V_i' we choose sets $U_{0,i}, \dots, U_{m-1,i} \in \mathfrak{U}$ such that

$$U_{0,i} \cap f^{-1}U_{1,i} \cap \cdots \cap f^{-m+1}U_{m-1,i} \supset V_i'.$$

For $s \geq r$, we write $s = r + mp + q$ with $p \geq 0$ and $0 \leq q < m$. Take i with $f^{r+mp}(x) \in V_i'$ and let $U_s = U_{q,i}$. Writing

$$a_p = U_{0,i} U_{1,i} \cdots U_{m-1,i},$$

we have

$$U = U_0 \cdots U_{r-1} a_0 a_1 \cdots a_{k_j - 1}.$$

For $a = (a_0, a_1, \ldots, a_{k_j-1})$, the values $\mu_a(e)$ of the measure μ_a defined by (4.28) in $\mathcal{U}^m$ has values $p_{1,r}, \ldots, p_{t,r}$ and some zeros. Therefore, by (4.31),

$$\frac{1}{m} H(a) = \frac{1}{m} \sum_{i=1}^{t} \chi(p_{i,r}) \leq h_\mu(f) + \varepsilon.$$

This establishes the second property in the lemma. The first property follows readily from (4.27), provided that j is sufficiently large. □

Given $\delta > 0$, $m \in \mathbb{N}$, and $u \in \mathbb{R}$, we denote by $Z_{m,u}$ the set of points $x \in \mathcal{L}(Z)$ such that the two properties in Lemma 4.3.2 hold for some measure $\mu \in V(x) \cap \mathcal{M}_f(Z)$, and

$$\int_Z \psi \, d\mu \in [u - \delta, u + \delta].$$

Moreover, we denote by b_N the number of vectors $U \in \mathcal{W}_N(\mathcal{U})$ satisfying the two properties in Lemma 4.3.2 for some point $x \in Z_{m,u}$.

Lemma 4.3.3. *For all sufficiently large N, we have*

$$b_N \leq \exp[N(h_\mu(f|Z) + 2\delta)].$$

Proof of the lemma. We proceed in a similar manner to that in the proof of Lemma 2.16 in [39]. Let us consider again the quantities $\mu_a(e)$ and $H(a)$ in (4.28) and (4.29). For each probability measure ν in E, $k \in \mathbb{N}$, and $\alpha \in (0,1)$, we define

$$R_k(\nu) = \big\{ a \in E^k : |\mu_a(e) - \nu(e)| < \alpha \text{ for every } e \in E \big\}.$$

Now let μ the Bernoulli measure in $E^{\mathbb{N}}$ with probabilities

$$p(e) = (1 - \alpha)\nu(E) + \alpha / \operatorname{card} E.$$

Given $a \in R_k(\nu)$, and the corresponding cylinder set $C_a \subset E^{\mathbb{N}}$, since each element $e \in E$ occurs in a at most $k(\nu(e) + \alpha)$ times, we have

$$\mu(C_a) \geq \prod_e p(e)^{k(\nu(e)+\alpha)}.$$

Moreover, since the cylinder sets C_a are pairwise disjoint and their union has measure 1, we obtain

$$1 \geq \operatorname{card} R_k(\nu) \prod_e p(e)^{k(\nu(e)+\alpha)},$$

and hence,

$$\begin{aligned}\frac{1}{k}\log\operatorname{card}R_k(\nu) &\leq \sum_e -(\nu(e)+\alpha)\log p(e)\\ &\leq H(\mu)-\sum_e 3\alpha\log p(e)\\ &\leq H(\mu)+3\alpha\operatorname{card}E\big(\log\operatorname{card}E-\log\alpha\big),\end{aligned}\tag{4.32}$$

since $\mu(e) \geq \alpha/\operatorname{card}E$. Letting $\alpha \to 0$, the second term in the right-hand side of (4.32) tends to 0, while $H(\mu) \to H(\nu)$ uniformly in μ. Therefore, for each $\varepsilon > 0$ there exists α sufficiently small such that

$$\frac{1}{k}\log\operatorname{card}R_k(\nu) \leq H(\nu)+\varepsilon \tag{4.33}$$

for every k and ν. Given $c > 0$, let A be the set of all probability measures ν in E such that $H(\nu) \leq c$, with the property that whenever $H(\nu') \leq c$ for some probability measure ν', there exists $\nu \in A$ such that

$$|\nu'(e)-\nu(e)| < \alpha \quad \text{for every} \quad e \in E$$

(with α as in (4.33)). Then

$$B_k := \big\{a \in E^k : H(a) \leq c\big\} \subset \bigcup_{\nu\in A} R_k(\nu),$$

and hence, by (4.33),

$$\frac{1}{k}\log\operatorname{card}B_k \leq \frac{1}{k}\log\operatorname{card}A + c + \varepsilon.$$

Therefore,

$$\limsup_{k\to\infty}\frac{1}{k}\log\operatorname{card}B_k \leq c+\varepsilon,$$

and letting $\varepsilon \to 0$ yields

$$\limsup_{k\to\infty}\frac{1}{k}\log\operatorname{card}B_k \leq c. \tag{4.34}$$

On the other hand, by the second property in Lemma 4.3.2, we have

$$b_N \leq (\operatorname{card}\mathcal{U})^m \operatorname{card}\big\{V \in (\mathcal{U}^m)^k : H(V) \leq m(h_\mu(f|Z)+\delta)\big\}. \tag{4.35}$$

Setting $c = m(h_\mu(f|Z)+\delta)$, the desired result follows now readily from (4.35) together with (4.34). $\square$

We proceed with the proof of the theorem. For each $l \in \mathbb{N}$, the collection of vectors U satisfying the two properties in Lemma 4.3.2 for some $x \in Z_{m,u}$ and $N \ge l$ cover the set $Z_{m,u}$. Therefore,

$$\begin{aligned} & M_{Z_{m,u}}(\alpha, \Phi, \mathcal{U}) \\ & \le \limsup_{l\to\infty} \sum_{N=l}^{\infty} b_N \exp\left[-\alpha N + N\left(\int_Z \psi\, d\mu + 2\delta\right) + \gamma_N(\Phi, \mathcal{U})\right] \\ & \le \limsup_{l\to\infty} \sum_{N=l}^{\infty} \exp\left[N\biggl(h_\mu(f|Z) + \int_Z \psi\, d\mu + 5\delta - \alpha + \limsup_{n\to\infty} \frac{\gamma_n(\Phi, \mathcal{U})}{n}\biggr)\right] \\ & \le \lim_{l\to\infty} \frac{\beta^l}{1-\beta}, \end{aligned} \tag{4.36}$$

where

$$\beta = \exp\biggl(-\alpha + c + \limsup_{n\to\infty} \frac{\gamma_n(\Phi, \mathcal{U})}{n} + 5\delta\biggr)$$

and

$$c = \sup_{\mu \in \mathcal{M}_f(Z)} \biggl(h_\mu(f|Z) + \int_Z \psi\, d\mu\biggr).$$

For

$$\alpha > c + 6\delta + \limsup_{n\to\infty} \frac{\gamma_n(\Phi, \mathcal{U})}{n}, \tag{4.37}$$

we have $\beta < e^{-\delta} < 1$, and it follows from (4.36) that

$$M_{Z_{m,u}}(\alpha, \Phi, \mathcal{U}) = 0 \quad \text{and} \quad \alpha > P_{Z_{m,u}}(\Phi, \mathcal{U}). \tag{4.38}$$

Now we take points $u_1, \ldots, u_r$ such that for each $u \in [-\|\psi\|_\infty, \|\psi\|_\infty]$, there exists $j \in \{1, \ldots, r\}$ with $|u - u_j| < \delta$. Then

$$\mathcal{L}(Z) = \bigcup_{m\in\mathbb{N}} \bigcup_{i=1}^{r} Z_{m,u_i},$$

and it follows from (4.37), (4.38), and the third property in Theorem 4.2.1 that

$$\begin{aligned} c + \lim_{\operatorname{diam}\mathcal{U}\to 0} \limsup_{n\to\infty} \frac{\gamma_n(\Phi, \mathcal{U})}{n} + 6\delta & \ge \lim_{\operatorname{diam}\mathcal{U}\to 0} \sup_{m,i} P_{Z_{m,u_i}}(\Phi, \mathcal{U}) \\ & = \lim_{\operatorname{diam}\mathcal{U}\to 0} P_{\mathcal{L}(Z)}(\Phi, \mathcal{U}) = P_{\mathcal{L}(Z)}(\Phi). \end{aligned}$$

Since δ is arbitrary, and the sequence Φ has the tempered variation property in (4.2), we obtain $c \ge P_{\mathcal{L}(Z)}(\Phi)$.

Now we establish the reverse inequality. We first prove an auxiliary result.

Lemma 4.3.4. *For each measure $\mu \in \mathcal{M}_f(Z)$, there exists an f-invariant function $\tilde{\psi} \in L^1(X,\mu)$ such that*

$$\lim_{n\to\infty} \frac{\varphi_n}{n} = \tilde{\psi}$$

μ-almost everywhere and in $L^1(X,\mu)$.

Proof of the lemma. Clearly, $\psi \in L^1(X,\mu)$, and

$$\lim_{n\to\infty} (\varphi_{n+1} - \varphi_n \circ f) = \psi$$

μ-almost everywhere and in $L^1(X,\mu)$. It thus follows from Theorem 1.4.8 that

$$\lim_{n\to\infty} \frac{1}{n}\left(\varphi_{n+1} - \varphi_1 \circ f^n - \sum_{k=0}^{n-1} \psi \circ f^k\right) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \psi_{n-k} \circ f^k = 0 \tag{4.39}$$

μ-almost everywhere and in $L^1(X,\mu)$, with ψ_n as in (4.26). By Birkhoff's ergodic theorem, there exists an f-invariant function $\tilde{\psi} \in L^1(X,\mu)$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \psi \circ f^k = \tilde{\psi}$$

μ-almost everywhere and in $L^1(X,\mu)$. Since φ_1 is bounded, the desired statement follows now readily from (4.39). □

The following statement gives a lower bound for the pressure for $P_Z(\Phi)$.

Lemma 4.3.5. *For each ergodic measure $\mu \in \mathcal{M}_f(Z)$, we have*

$$P_Z(\Phi) \geq h_\mu(f|Z) + \int_Z \psi \, d\mu.$$

Proof of the lemma. We first observe that given $\varepsilon > 0$, there exist $\delta \in (0,\varepsilon)$, a measurable partition $\xi = \{C_1, \ldots, C_m\}$, and a finite open cover $\mathcal{U} = \{U_1, \ldots, U_k\}$ of X, with $k \geq m$, such that:

1. $\operatorname{diam} U_i \leq \varepsilon$ and $\operatorname{diam} C_j \leq \varepsilon$ for each i and j;
2. $\overline{U_i} \subset C_i$ and $\mu(C_i \setminus U_i) < \delta$ for $i = 1, \ldots, m$;
3. $\mu\left(\bigcup_{i=m+1}^k U_i\right) < \delta$ and $2\delta \log m \leq \varepsilon$.

For each $x \in Z$ and $n \in \mathbb{N}$, let $t_n(x)$ be the number of integers $l \in [0,n)$ such that $f^l(x) \in U_i$ for some $i \in \{m+1, \ldots, k\}$. It follows from Birkhoff's ergodic theorem and Egorov's theorem that there exist $N_1 \in \mathbb{N}$ and a measurable set $A_1 \subset Z$ with $\mu(A_1) \geq 1 - \delta$ such that

$$\frac{t_n(x)}{n} < 2\delta \quad \text{and} \quad \left|\frac{\varphi_n(x)}{n} - \int_Z \psi \, d\mu\right| < \delta \tag{4.40}$$

for every $x \in A_1$ and $n > N_1$. Moreover, writing

$$\eta_n = \bigvee_{j=0}^{n} f^{-j}(\xi|Z),$$

where $\xi|Z$ is the partition of Z induced by ξ, it follows from the Shannon–McMillan–Breiman theorem and again Egorov's theorem that there exist $N_2 \in \mathbb{N}$ and a measurable set $A_2 \subset Z$ with $\mu(A_2) \geq 1 - \delta$ such that

$$\mu(\eta_n(x)) \leq \exp\left[(-h_\mu(f|Z, \xi|Z) + \delta)n\right] \tag{4.41}$$

for every $x \in A_2$ and $n > N_2$. We set $N = \max\{N_1, N_2\}$ and $A = A_1 \cap A_2$, and we note that $\mu(A) \geq 1 - 2\delta$.

We proceed with the proof of the lemma. Given $\alpha \in \mathbb{R}$, there exists $N' \geq N$ such that for each $n \geq N'$, there exists a collection $\Gamma \subset \bigcup_{m\geq n} \mathcal{W}_m(\mathcal{U})$ covering Z with

$$\left|\sum_{U\in\Gamma} \exp\left(-\alpha m(U) + \varphi(U)\right) - M_Z(\alpha, \Phi, \mathcal{U})\right| < \delta. \tag{4.42}$$

Given $l \in \mathbb{N}$, let

$$\Gamma_l = \left\{U \in \Gamma : m(U) = l, X(U) \cap A \neq \varnothing\right\},$$

and define $X_l = \bigcup_{U\in\Gamma_l} X(U)$. Now we proceed in a similar manner to that in the proof of Lemma 2 in [153] to show that

$$\operatorname{card}\Gamma_l \geq \mu(X_l \cap A) \exp\left[h_\mu(f|Z, \xi|Z)l - (1 + 2\log\operatorname{card}\xi)\delta l\right] \tag{4.43}$$

for each $l \in \mathbb{N}$. For this, let L_l be the number of elements C of the partition η_l such that $C \cap X_l \cap A \neq \varnothing$. Summing over these elements we obtain

$$\sum_{C\cap X_l\cap A\neq\varnothing} \mu(C) \geq \mu(X_l \cap A). \tag{4.44}$$

On the other hand, since $C \cap A_2 \neq \varnothing$, it follows from (4.41) and (4.44) that

$$L_l \geq \mu(X_l \cap A) \exp\left[(h_\mu(f|Z, \xi|Z) - \delta)l\right]. \tag{4.45}$$

Moreover, given $U \in \Gamma_l$, since $X(U) \cap A_1 \neq \varnothing$, it follows from the first inequality in (4.40) that the number S_U of the elements C of the partition η_l such that $X(U) \cap C \cap A \neq \varnothing$ satisfies

$$S_U \leq m^{2\delta l} = \exp(2\delta l \log m). \tag{4.46}$$

Inequality (4.43) follows now readily from (4.45) and (4.46).

We continue to consider the collection Γ in (4.42). By (4.40), we have

$$\sup_{X(U)} \varphi_l \geq l\left(\int_Z \psi \, d\mu - \delta\right) - \gamma_l(\Phi, \mathcal{U})$$

for each $l \ge N'$ and $U \in \Gamma_l$. Therefore,

$$\begin{aligned}
&\sum_{U\in\Gamma} \exp\bigl(-\alpha m(U) + \varphi(U)\bigr)\\
&\ge \sum_{l=N'}^{\infty} \sum_{U\in\Gamma_l} \exp\biggl(- \alpha l + \sup_{X(U)} \varphi_l\biggr)\\
&\ge \sum_{l=N'}^{\infty} \operatorname{card}\Gamma_l \times \exp\biggl[\biggl(- \alpha + \int_Z \psi \, d\mu - \delta\biggr) l - \gamma_l(\Phi, \mathfrak{U})\biggr]\\
&\ge \sum_{l=N'}^{\infty} \mu(X_l \cap A)\\
&\quad \times \exp\biggl[\biggl(h_\mu(f|Z, \xi|Z) + \int_Z \psi \, d\mu - \frac{\gamma_l(\Phi, \mathfrak{U})}{l} - \alpha\biggr) l - 2(1 + \log \operatorname{card}\xi)\delta l\biggr].
\end{aligned}$$

For each finite open cover $\mathfrak{U}$ of X, we write

$$\alpha(\mathfrak{U}) = h_\mu(f|Z, \xi|Z) + \int_Z \psi \, d\mu - \limsup_{n\to\infty} \frac{\gamma_n(\Phi, \mathfrak{U})}{n},$$

and given $\alpha < \alpha(\mathfrak{U})$, let us take $\delta > 0$ such that

$$\alpha(\mathfrak{U}) - \alpha - 2(1 + \log \operatorname{card}\xi)\delta > \delta.$$

Without loss of generality, we also assume that N' is sufficiently large so that

$$\frac{\gamma_l(\Phi, \mathfrak{U})}{l} \le \limsup_{n\to\infty} \frac{\gamma_n(\Phi, \mathfrak{U})}{n} + \delta$$

for every $l \ge N'$. This implies that

$$\sum_{U\in\Gamma} \exp\bigl(-\alpha m(U) + \varphi(U)\bigr) \ge \sum_{l=N'}^{\infty} \mu(X_l \cap A) = \mu(A) \ge 1 - 2\delta,$$

for the collection Γ in (4.42), and thus,

$$M_Z(\alpha, \Phi, \mathfrak{U}) > 1 - 3\delta > 0$$

for all sufficiently small δ. Therefore, $P_Z(\Phi, \mathfrak{U}) \ge \alpha$ and $P_Z(\Phi, \mathfrak{U}) \ge \alpha(\mathfrak{U})$.

Now we consider sequences $(\xi_l)_{l\in\mathbb{N}}$ of measurable partitions and $(\mathfrak{U}_l)_{l\in\mathbb{N}}$ of finite open covers, as in the beginning of the proof of the lemma, with $\varepsilon = 1/l$. Since $\operatorname{diam}\xi_l \to 0$ when $l \to \infty$, it follows from (1.10) that

$$\lim_{l\to\infty} h_\mu(f|Z, \xi_l|Z) = h_\mu(f|Z).$$

By Theorem 4.1.2 and the tempered variation property in (4.2) we obtain

$$\begin{aligned}
P_Z(\Phi) &= \lim_{l\to\infty} P_Z(\Phi, \mathcal{U}_l) \\
&\geq \limsup_{l\to\infty} \alpha(\mathcal{U}_l) \\
&= \lim_{l\to\infty} h_\mu(f|Z, \xi_l|Z) + \int_Z \psi\, d\mu - \lim_{l\to\infty} \limsup_{n\to\infty} \frac{\gamma_n(\Phi, \mathcal{U}_l)}{n} \\
&= h_\mu(f|Z) + \int_Z \psi\, d\mu.
\end{aligned}$$

This completes the proof of the lemma. □

To complete the proof of the theorem, for each $\mu \in \mathcal{M}_f$ we consider the set

$$Z_\mu = \big\{x \in Z : V(x) = \{\mu\}\big\}. \tag{4.47}$$

One can easily verify that Z_μ is f-invariant and measurable. If the measure $\mu \in \mathcal{M}_f(Z)$ is ergodic, then Z_μ is a nonempty subset of $\mathcal{L}(Z)$ with $\mu(Z_\mu) = 1$. By the second property in Theorem 4.2.1 and Lemmas 4.3.4 and 4.3.5, we obtain

$$P_{\mathcal{L}(Z)}(\Phi) \geq P_{Z_\mu}(\Phi) \geq h_\mu(f|Z_\mu) + \int_{Z_\mu} \psi\, d\mu = h_\mu(f|Z) + \int_Z \psi\, d\mu.$$

When the measure $\mu \in \mathcal{M}_f(Z)$ is arbitrary, we first decompose the space X into ergodic components. Then the previous result for ergodic measures together with standard arguments from ergodic theory can be used to show that

$$P_{\mathcal{L}(Z)}(\Phi) > \sup_{\mu\in\mathcal{M}_f(Z)} \left(h_\mu(f|Z) + \int_Z \psi\, d\mu\right).$$

This completes the proof of the theorem. □

We emphasize that the set Z in Theorem 4.3.1 need not be compact. Moreover, since each measure $\mu \in \mathcal{M}_f(Z)$ is concentrated on Z, we have

$$h_\mu(f|Z) = h_\mu(f) \quad \text{and} \quad \int_Z \psi\, d\mu = \int_X \psi\, d\mu.$$

Therefore, identity (4.25) can be written in the form

$$P_{\mathcal{L}(Z)}(\Phi) = \sup_{\mu\in\mathcal{M}_f(Z)} \left(h_\mu(f) + \int_X \psi\, d\mu\right).$$

Theorem 4.3.1 has the following immediate consequences.

Corollary 4.3.6. *Under the hypotheses of Theorem* 4.3.1:

1. *if* $V(x) \cap \mathcal{M}_f(Z) \neq \varnothing$ *for each* $x \in Z$, *then*

$$P_Z(\Phi) = \sup_{\mu \in \mathcal{M}_f(Z)} \left(h_\mu(f|Z) + \int_Z \psi \, d\mu \right);$$

2. *if* $\mu \in \mathcal{M}_f(Z)$ *is ergodic, then*

$$P_{Z_\mu}(\Phi) = h_\mu(f|Z) + \int_Z \psi \, d\mu,$$

with the set Z_μ *as in* (4.47).

When the set Z is compact and f-invariant, we have $V(x) \cap \mathcal{M}_f(Z) \neq \varnothing$ for each $x \in Z$. However, this may not hold for an arbitrary set Z (see [153] for an example in the additive setting). We observe that identity (4.25) can be written in the form $P_{\mathcal{L}(Z)}(\Phi) = P_Z(\Psi)$, where Ψ is the sequence of functions

$$\psi_n = \sum_{k=0}^{n-1} \psi \circ f^k.$$

Theorem 4.3.1 and Corollary 4.3.6 are nonadditive versions of the classical variational principle in Theorem 2.3.1. For compact sets, they generalize the variational principle obtained by Ruelle in [164] for expansive maps, and by Walters in [194] in the general case. For arbitrary sets, they generalize the variational principle established by Pesin and Pitskel' in [153]. When Φ is the sequence of functions identically zero, we recover the variational principle for the topological entropy due to Goodwyn [80] (showing that the topological entropy is an upper bound for the metric entropy), Dinaburg [48] (for a finite-dimensional space X), and Goodman [78] (for an arbitrary space), as well as the variational principle for the topological entropy of arbitrary sets due to Bowen [37] (see [153] for details).

4.4 Bowen's equation

As we already mentioned, the unique solution s of Bowen's equation $P(s\varphi) = 0$, where φ is a certain function associated to an invariant set, is often related to the Hausdorff dimension of the set. As we shall see in later chapters, certain appropriate nonadditive versions of Bowen's equation play an important role in the dimension theory and in the multifractal analysis of dynamical systems. Here we present natural assumptions under which these equations have unique solutions.

The following result considers appropriate nonadditive versions of Bowen's equation. For each $s \in \mathbb{R}$ we denote by $s\Phi$ the sequence of functions $s\varphi_n$.

Theorem 4.4.1 ([5]). *Assume that* $h(f) < \infty$, *and that there exist* $K_1, K_2 < 0$ *such that*

$$K_1 n \leq \varphi_n \leq K_2 n \quad \text{for every} \quad n \in \mathbb{N}.$$

Then the following properties hold:

1. *the functions* $s \mapsto P_Z(s\Phi)$*,* $s \mapsto \underline{P}_Z(s\Phi)$*, and* $s \mapsto \overline{P}_Z(s\Phi)$ *are strictly decreasing and Lipschitz;*
2. *there exist unique roots* s_P*,* $s_{\underline{P}}$*, and* $s_{\overline{P}}$ *respectively of the equations*

$$P_Z(s\Phi) = 0, \quad \underline{P}_Z(s\Phi) = 0, \quad \text{and} \quad \overline{P}_Z(s\Phi) = 0, \tag{4.48}$$

which satisfy

$$0 \le s_P \le s_{\underline{P}} \le s_{\overline{P}} < \infty. \tag{4.49}$$

Proof. We first show that the pressure functions are finite. For each $s \ge 0$ we have

$$M_Z(\alpha - sK_1, 0, \mathfrak{U}) \le M_Z(\alpha, s\Phi, \mathfrak{U}) \le M_Z(\alpha - sK_2, 0, \mathfrak{U}),$$

and hence,

$$h(f|Z) + sK_1 \le P_Z(s\Phi) \le h(f|Z) + sK_2.$$

Similarly, for each $s \le 0$ we have

$$h(f|Z) + sK_2 \le P_Z(s\Phi) \le h(f|Z) + sK_1.$$

Since $h(f|Z) \le h(f) < \infty$, the number $P_Z(s\Phi)$ is finite for each $s \in \mathbb{R}$. Similar arguments show that $\underline{P}_Z(s\Phi)$ and $\overline{P}_Z(s\Phi)$ are finite for each $s \in \mathbb{R}$. Since

$$|\varphi_n(x)| \le \max\{-K_1, K_2\}n = -K_1 n$$

for each $x \in X$ and $n \in \mathbb{N}$, it follows readily from the second property in Theorem 4.2.2 that the three functions in the first statement of the theorem are Lipschitz with Lipschitz constant $-K_1$. Moreover, proceeding as in the proof of Theorem 4.2.2, for $s' \ge s$ we have

$$-(s' - s)K_2 \le P_Z(s\Phi) - P_Z(s'\Phi) \le -(s' - s)K_1. \tag{4.50}$$

This shows that the function $s \mapsto P_Z(s\Phi)$ is strictly decreasing, and thus, there is a unique root s_P of the equation $P_Z(s\Phi) = 0$. Similar arguments show that there exist unique roots of the other two equations in (4.48). Moreover, since the three functions in the first statement of the theorem are strictly decreasing, the first property in Theorem 4.2.1 yields inequality (4.49). Setting $s = 0$ in (4.50) we obtain

$$-s'K_2 \le h(f|Z) - P_Z(s'\Phi) \le -s'K_1.$$

Since $h(f|Z) \ge 0$, we have $s_P \ge 0$. To show that $s_{\overline{P}} < \infty$, we observe that

$$\overline{h}(f|Z) + sK_1 \le \overline{P}_Z(s\Phi) \le \overline{h}(f|Z) + sK_2$$

for each $s \ge 0$, and that

$$\overline{h}(f|Z) \le \overline{h}(f|X) = h(f) < \infty,$$

by the second properties in Theorems 4.2.1 and 4.2.6. □

Theorem 4.4.1 has a corresponding version when the functions φ_n are positive. Each equation in (4.48) is a nonadditive version of Bowen's equation. Introduced by Bowen in [40], it establishes the connection between the thermodynamic formalism and the dimension theory of dynamical systems. Namely, its unique root often gives the exact value or is a good estimate for the dimension.

More generally, one can consider parameterized families Φ_s of sequences of functions. A slight modification of the proof of Theorem 4.4.1 yields the following statement.

Theorem 4.4.2. *Assume that $h(f) < \infty$, and that there exist $K_1, K_2 < 0$ such that the family Φ_s, for $s \in \mathbb{R}$, satisfies*

$$K_1(t-s) \le \frac{\varphi_{t,n} - \varphi_{s,n}}{n} \le K_2(t-s)$$

for every $n \in \mathbb{N}$ and $t, s \in \mathbb{R}$ with $t > s$. Then the following properties hold:

1. *the functions $s \mapsto P_Z(\Phi_s)$, $s \mapsto \underline{P}_Z(\Phi_s)$, and $s \mapsto \overline{P}_Z(\Phi_s)$ are strictly decreasing and Lipschitz;*
2. *there exist unique roots s_P^*, $s_{\underline{P}}^*$, and $s_{\overline{P}}^*$ respectively of the equations*

$$P_Z(\Phi_s) = 0, \quad \underline{P}_Z(\Phi_s) = 0, \quad \text{and} \quad \overline{P}_Z(\Phi_s) = 0,$$

which satisfy

$$0 \le s_P^* \le s_{\underline{P}}^* \le s_{\overline{P}}^* < \infty.$$

4.5 The case of symbolic dynamics

We consider in this section the particular case of symbolic dynamics, and we obtain explicit formulas for the capacity topological pressures in terms of cylinder sets (or equivalently in terms of finite admissible sequences). Together with Theorem 4.2.6, these formulas play an important role in the applications of Chapters 5 and 8.

Let $\sigma\colon X \to X$ be the shift map in the set $X = \Sigma_\kappa^+$ of one-sided sequences of κ symbols (see Section 3.1 for the definitions). We denote by $\mathcal{U}_n$ the (finite) open cover of Σ_κ^+ formed by the n-cylinder sets. Clearly, $\operatorname{diam} \mathcal{U}_n \to 0$ when $n \to \infty$. We also consider a family of positive numbers $a_{i_1 \cdots i_n}$ for each $(i_1 i_2 \cdots) \in \Sigma_\kappa^+$ and $n \in \mathbb{N}$. We denote by Φ_a the sequence of functions $\varphi_{a,n}\colon \Sigma_\kappa^+ \to \mathbb{R}$ defined by

$$\varphi_{a,n}(i_1 i_2 \cdots) = \log a_{i_1 \cdots i_n}.$$

Since $\varphi_{a,n}$ is constant on the set $X(U)$ for every $n \in \mathbb{N}$ and $U \in \mathcal{W}_n(\mathcal{U}_1)$, the sequence Φ_a has tempered variation (see (4.2)).

Now let $Q \subset \Sigma_\kappa^+$ be an arbitrary set (in particular, it need not be compact neither shift-invariant). For each $n \in \mathbb{N}$, we denote by Q_n the set of all *Q-admissible sequences* of length n, that is, all sequences $(i_1 \cdots i_n)$ such that

$$(i_1 \cdots i_n) = (j_1 \cdots j_n) \quad \text{for some} \quad (j_1 j_2 \cdots) \in Q.$$

Given $s \in \mathbb{R}$, the following statement gives an equivalent description of the capacity pressures for the sequence of functions $s\Phi_a$ in the set Q (with respect to the shift map).

Theorem 4.5.1 ([5]). *For every $s \in \mathbb{R}$ and $l \in \mathbb{N}$, we have*

$$\underline{P}_Q(s\Phi_a) = \underline{P}_Q(s\Phi_a, \mathcal{U}_l) = \liminf_{n\to\infty} \frac{1}{n} \log \sum_{\gamma \in Q_n} a_\gamma^s \tag{4.51}$$

and

$$\overline{P}_Q(s\Phi_a) = \overline{P}_Q(s\Phi_a, \mathcal{U}_l) = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{\gamma \in Q_n} a_\gamma^s. \tag{4.52}$$

Proof. Let $X = \Sigma_\kappa^+$. We observe that there is a unique collection $\Gamma \subset \mathcal{W}_n(\mathcal{U}_l)$ covering Σ_κ^+: indeed, for each $U \in \mathcal{W}_n(\mathcal{U}_l)$ the set $X(U)$ is an $(n+l-1)$-cylinder set, and $X(U) \cap X(V) = \varnothing$ whenever $U \neq V$. Therefore,

$$\mathcal{Z}_n(Q, s\Phi_a, \mathcal{U}) = \sum_{(i_1\cdots i_{n+l-1})\in Q_{n+l-1}} a_{i_1\cdots i_n}^s, \tag{4.53}$$

where the function $\mathcal{Z}_n$ is defined in (4.13). For each $l \in \mathbb{N}$, we have

$$\begin{aligned}\kappa^{-l+1} \sum_{(i_1\cdots i_{n+l-1})\in Q_{n+l-1}} a_{i_1\cdots i_n}^s &\le \sum_{(i_1\cdots i_n)\in Q_n} a_{i_1\cdots i_n}^s \\ &\le \sum_{(i_1\cdots i_{n+l-1})\in Q_{n+l-1}} a_{i_1\cdots i_n}^s,\end{aligned}$$

and by (4.53), this is the same as

$$\kappa^{-l+1}\mathcal{Z}_n(Q, s\Phi_a, \mathcal{U}_l) \le \mathcal{Z}_n(Q, s\Phi_a, \mathcal{U}_1) \le \mathcal{Z}_n(Q, s\Phi_a, \mathcal{U}_l).$$

By (4.14), we obtain

$$\underline{P}_Q(s\Phi_a, \mathcal{U}_l) = \underline{P}_Q(s\Phi_a, \mathcal{U}_1)$$

and hence,

$$\begin{aligned}\underline{P}_Q(s\Phi_a) &= \lim_{l\to\infty} \underline{P}_Q(s\Phi_a, \mathcal{U}_l) \\ &= \underline{P}_Q(s\Phi_a, \mathcal{U}_1) \\ &= \liminf_{n\to\infty} \frac{1}{n} \log \mathcal{Z}(Q, s\Phi_a, \mathcal{U}_1).\end{aligned} \tag{4.54}$$

Identity (4.51) follows now readily from (4.53) and (4.54). Using (4.15), a similar argument establishes identity (4.52). □

The following example taken from [5] illustrates that the numbers $s_{\underline{P}}$ and $s_{\overline{P}}$ given by Theorem 4.4.1 may not coincide, even for a sequence of constant functions.

Example 4.5.2. There exist constant functions $\varphi_n \colon \Sigma_2^+ \to \mathbb{R}$ such that:

1. the sequence $\Phi = (\varphi_n)_{n\in\mathbb{N}}$ satisfies the hypotheses of Theorem 4.4.1;
2. there exist constants $a, b > 0$ such that $a \le \varphi_{n+1} - \varphi_n \le b$ for every $n \in \mathbb{N}$;
3. the sequence Φ is not subadditive;
4. the unique roots $s_{\underline{P}}$ and $s_{\overline{P}}$ of the equations $\underline{P}_{\Sigma_2^+}(s\Phi) = 0$ and $\overline{P}_{\Sigma_2^+}(s\Phi) = 0$ satisfy $s_{\underline{P}} < s_{\overline{P}}$.

Construction. Given $a > b > 0$, let us take $\delta > 0$ such that $\delta < (a-b)/2$. Now we define inductively a sequence of positive integers m_n. Given $k \in \mathbb{N}$, we shall write

$$n_k(a,b) = a \sum_{j \text{ odd } \le k} m_j + b \sum_{j \text{ even} \le k} m_j,$$

and $n_k = m_1 + \cdots + m_k$. Fix $m_1 \in \mathbb{N}$. For each $k \in \mathbb{N}$, we choose $m_k \in \mathbb{N}$ such that

$$|n_k(a,b)/n_k - a| < \delta/k \text{ if } k \text{ is odd,}$$

and

$$|n_k(a,b)/n_k - b| < \delta/k \text{ if } k \text{ is even.}$$

For each $n \in \mathbb{N}$ and $\omega \in \Sigma_2^+$, we define

$$\varphi_n(\omega) = \lambda_n = -n_k(a,b) - \begin{cases} a(n-n_k), & k \text{ odd,} \\ b(n-n_k), & k \text{ even,} \end{cases}$$

where k is the largest positive integer such that $n_k < n$. One can easily verify that the sequence Φ satisfies properties 1 and 2. Since $\varphi_{n_k} > n_k\varphi_1$ for every $k > 1$, the sequence Φ is not subadditive.

For the last property, we note that

$$\begin{aligned} \frac{1}{n} \log \sum_{i_1 \cdots i_n} \exp\left(s \sup_{C_{i_1 \cdots i_n}} \varphi_n \right) &= \frac{1}{n} \log\left(2^n \exp(s\lambda_n)\right) \\ &= \log 2 + \frac{s\lambda_n}{n} \end{aligned}$$

for each $s \in \mathbb{R}$, where $C_{i_1 \cdots i_n}$ is an n-cylinder set. Since $\lambda_{n_k} = -n_k(a,b)$, it follows from Theorem 4.5.1 that

$$\begin{aligned} \underline{P}_{\Sigma_2^+}(s\Phi) &= \log 2 + s \liminf_{n\to\infty} \frac{\lambda_n}{n} \\ &\le \log 2 - s \limsup_{k\to\infty} \frac{n_k(a,b)}{n_k} \le \log 2 - sa, \end{aligned}$$

and

$$\begin{aligned}\overline{P}_{\Sigma_2^+}(s\Phi) &= \log 2 + s \limsup_{n\to\infty} \frac{\lambda_n}{n} \\ &\geq \log 2 - s \liminf_{k\to\infty} \frac{n_k(a,b)}{n_k} \geq \log 2 - sb,\end{aligned}$$

for each $s \geq 0$. Since $-an \leq \lambda_n \leq -bn$ for every $n \in \mathbb{N}$, the previous inequalities imply that

$$\underline{P}_{\Sigma_2^+}(s\Phi) = \log 2 - sa \quad \text{and} \quad \overline{P}_{\Sigma_2^+}(s\Phi) = \log 2 - sb$$

for every $s \in \mathbb{R}$. Therefore, $s_{\underline{P}} = \log 2/a < \log 2/b = s_{\overline{P}}$. □

Chapter 5

Dimension Estimates for Repellers

We consider in this chapter the dimension of repellers, which are invariant sets of a hyperbolic noninvertible dynamics. After describing how Markov partitions can be used to model repellers, we present several applications of the nonadditive thermodynamic formalism to the study of their dimension. In particular, we establish lower and upper dimension estimates for a large class of repellers. Among other results, as a simple corollary of this approach we obtain a new proof of the corresponding result in the case of conformal dynamics (in which case the lower and upper dimension estimates coincide). We consider smooth expanding maps as well as their continuous counterpart, which represents a substantial extension of the theory. For completeness, at the end of the chapter we give a sufficiently broad panorama of the existing results concerning dimension estimates for repellers of smooth dynamical systems, with emphasis on the relation to the thermodynamic formalism. Among other topics, we consider self-affine repellers, their nonlinear extensions, and repellers of nonuniformly expanding maps.

5.1 Repellers of continuous expanding maps

We consider in this section the general case of repellers of continuous expanding maps. In particular, we describe several applications of the nonadditive thermodynamic formalism to the study of the dimension of repellers in this general setting.

5.1.1 Basic notions

We first introduce some basic concepts, starting with the notions of continuous expanding map and of repeller.

Let (X, d) be a compact metric space.

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Definition 5.1.1. A continuous map $h\colon X \to X$ is said to be *expanding* if it is a local homeomorphism at every point, and there exist constants $a \ge b > 1$ and $r_0 > 0$ such that

$$B(h(x), br) \subset h(B(x, r)) \subset B(h(x), ar). \tag{5.1}$$

for every $x \in X$ and $0 < r < r_0$. We then say that X is a *repeller* of h.

Any repeller of a continuous expanding map h is the union of a finite number of disjoint compact sets $X_1, \dots, X_m$ that are cyclically permuted by h, and such that $h^m|X_i$ is topologically mixing for $i = 1, \dots, m$. We recall that a transformation $h\colon X \to X$ is said to be *topologically mixing* on a set $Y \subset X$ if for each open sets U and V with nonempty intersection with Y there exists $n \in \mathbb{N}$ such that

$$h^m(U) \cap V \cap Y \ne \varnothing \quad \text{for every} \quad m > n.$$

It follows from Proposition 5.1.4 that a continuous expanding map is locally bi-Lipschitz. Therefore,

$$\dim_H X = \max_{i=1,\dots,m} \dim_H X_i = \dim_H X_i$$

for $i = 1, \dots, m$, with analogous identities for the lower and upper box dimensions. Hence, without loss of generality we may always assume that a continuous expanding map is topologically mixing.

Now we introduce the notion of Markov partition.

Definition 5.1.2. A finite cover of X by nonempty closed sets $R_1, \dots, R_\kappa$ is called a *Markov partition* of X (with respect to h) if:

1. $\overline{\operatorname{int} R_i} = R_i$ for each i;
2. $\operatorname{int} R_i \cap \operatorname{int} R_j = \varnothing$ whenever $i \ne j$;
3. $h(R_i) \supset R_j$ whenever $h(\operatorname{int} R_i) \cap \operatorname{int} R_j \ne \varnothing$.

Ruelle showed in [166] that any repeller of a continuous expanding map has Markov partitions of arbitrarily small diameter.

We also describe how Markov partitions can be use to model a repeller. Given a Markov partition $R_1, \dots, R_\kappa$, we consider the a $\kappa \times \kappa$ matrix $A = (a_{ij})$ with entries

$$a_{ij} = \begin{cases} 1 & \text{if } h(\operatorname{int} R_i) \cap \operatorname{int} R_j \ne \varnothing, \\ 0 & \text{if } h(\operatorname{int} R_i) \cap \operatorname{int} R_j = \varnothing. \end{cases} \tag{5.2}$$

We also consider the shift map $\sigma\colon \Sigma_\kappa^+ \to \Sigma_\kappa^+$ (see Section 3.1), and the (one-sided) topological Markov chain $\sigma|\Sigma_A^+$ with transition matrix A (see Definition 3.4.3). We note that if h is topologically mixing on the repeller X, then there exists $q \in \mathbb{N}$ such that A^q has only positive entries.

By (5.1), we can define a *coding map* $\chi\colon \Sigma_A^+ \to X$ of the repeller X by

$$\chi(i_1 i_2 \cdots) = \bigcap_{k=0}^{\infty} h^{-k} R_{i_{k+1}}.$$

The map χ is onto and is Hölder continuous with respect to the distance in Σ_κ^+ given by (3.1). Moreover, it satisfies

$$\chi \circ \sigma = h \circ \chi, \tag{5.3}$$

that is, we have the commutative diagram

$$\begin{array}{ccc} \Sigma_A^+ & \xrightarrow{\sigma} & \Sigma_A^+ \\ \chi \downarrow & & \downarrow \chi \\ X & \xrightarrow{h} & X \end{array}.$$

The map χ may not be invertible, although $\operatorname{card} \chi^{-1} x \le \kappa^2$ for every $x \in \Sigma_A^+$. Nevertheless, identity (5.3) still allows one to see χ as a dictionary between the symbolic dynamics $\sigma|\Sigma_A^+$ (as well as often the results at this level) and the dynamics of the map h in X.

For each $(i_1 i_2 \cdots) \in \Sigma_A^+$ and $n \in \mathbb{N}$, let

$$R_{i_1 \cdots i_n} = \bigcap_{k=0}^{n-1} h^{-k} R_{i_{k+1}}. \tag{5.4}$$

For each $n \in \mathbb{N}$, these sets intersect at most along their boundaries (due to the second condition in the definition of Markov partition). Given a continuous function $\varphi\colon X \to \mathbb{R}$, it follows from Theorem 3.1.1 that its (classical) topological pressure is given by

$$P(\varphi) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{(i_1 \cdots i_n) \in S_n} \exp \sup_{R_{i_1 \cdots i_n}} \sum_{k=0}^{n-1} \varphi \circ h^k,$$

where S_n is the set of all Σ_A^+-admissible sequences of length n, that is, the sequences $(i_1 \cdots i_n)$ formed by the first n elements of some sequence in Σ_A^+. We note that $P(\varphi)$ is independent of the particular Markov partition used to compute it.

Now we introduce families of numbers that shall be useful to obtain dimension estimates for repellers.

Definition 5.1.3. For each $\omega = (i_1 i_2 \cdots) \in \Sigma_A^+$ and $n, k \in \mathbb{N}$, we define the *lower* and *upper ratio coefficients* by

$$\underline{\lambda}_k(\omega, n) = \min \inf \left\{ \frac{d(x, y)}{d(h^n(x), h^n(y))} : x, y \in R_{j_1 \cdots j_{n+k}} \text{ and } x \neq y \right\} \tag{5.5}$$

and

$$\overline{\lambda}_k(\omega,n)=\max\sup\left\{\frac{d(x,y)}{d(h^n(x),h^n(y))}:x,y\in R_{j_1\cdots j_{n+k}}\text{ and }x\neq y\right\},\tag{5.6}$$

where the minimum and the maximum are taken over all Σ_A^+-admissible finite sequences $(j_1\cdots j_{n+k})$ such that $(j_1\cdots j_n)=(i_1\cdots i_n)$.

We observe that the sequences $k\mapsto\underline{\lambda}_k(\omega,n)$ and $k\mapsto\overline{\lambda}_k(\omega,n)$ are respectively nondecreasing and nonincreasing, for each given $\omega\in\Sigma_A^+$ and $n\in\mathbb{N}$.

Now we define sequences of functions $\underline{\varphi}_{k,n}$ and $\overline{\varphi}_{k,n}$ in Σ_A^+ by

$$\underline{\varphi}_{k,n}(\omega)=\log\underline{\lambda}_k(\omega,n)\quad\text{and}\quad\overline{\varphi}_{k,n}(\omega)=\log\overline{\lambda}_k(\omega,n),\tag{5.7}$$

and we denote them respectively by $\underline{\Phi}_k$ and $\overline{\Phi}_k$.

Proposition 5.1.4. *We have*

$$-n\log a\leq\underline{\varphi}_{k,n}(\omega)\leq\overline{\varphi}_{k,n}(\omega)\leq-n\log b$$

for each $\omega\in\Sigma_A^+$ and $n,k\in\mathbb{N}$, with k sufficiently large.

Proof. We start with two auxiliary results.

Lemma 5.1.5. *For each $\omega\in\Sigma_A^+$ and $n,m,k\in\mathbb{N}$, we have*

$$\underline{\lambda}_k(\omega,n+m)\geq\underline{\lambda}_k(\omega,n)\underline{\lambda}_k(\sigma^n(\omega),m)\tag{5.8}$$

and

$$\overline{\lambda}_k(\omega,n+m)\leq\overline{\lambda}_k(\omega,n)\overline{\lambda}_k(\sigma^n(\omega),m).\tag{5.9}$$

Proof of the lemma. Given $\omega=(i_1i_2\cdots)\in\Sigma_A^+$ and $n,m,k\in\mathbb{N}$, we denote by A_k the set of all Σ_A^+-admissible finite sequences $(j_1\cdots j_{n+m+k})$ (see Section 4.5 for the definition) such that $(j_1\cdots j_{n+m})=(i_1\cdots i_{n+m})$. Since

$$\frac{d(x,y)}{d(h^{n+m}(x),h^{n+m}(y))}=\frac{d(x,y)}{d(h^n(x),h^n(y))}\cdot\frac{d(h^n(x),h^n(y))}{d(h^{n+m}(x),h^{n+m}(y))},$$

we have

$$\begin{aligned}\underline{\lambda}_k(\omega,n+m)&=\min_{A_k}\inf\left\{\frac{d(x,y)}{d(h^{n+m}(x),h^{n+m}(y))}:x,y\in R\text{ and }x\neq y\right\}\\&\geq\min_{A_k}\inf\left\{\frac{d(x,y)}{d(h^n(x),h^n(y))}:x,y\in R\text{ and }x\neq y\right\}\\&\quad\times\min_{A_k}\inf\left\{\frac{d(z,w)}{d(h^m(z),h^m(w))}:z,w\in h^n(R)\text{ and }z\neq w\right\},\end{aligned}$$

where $R=R_{j_1\cdots j_{n+m+k}}$. Let us consider the last product. We can estimate from below the minimum in the first term by the minimum over all Σ_A^+-admissible finite sequences $(j_1 \cdots j_{n+m+k})$ such that $(j_1 \cdots j_n) = (i_1 \cdots i_n)$. For the second term, we observe that

$$h^n(R) = h^n(R_{j_1 \cdots j_{n+m+k}}) \subset R_{j_{n+1} \cdots j_{n+m+k}}.$$

Hence,

$$\begin{aligned}\underline{\lambda}_k(\omega, n+m) &\geq \underline{\lambda}_{m+k}(\omega, n)\underline{\lambda}_k(\sigma^n(\omega), m) \\ &\geq \underline{\lambda}_k(\omega, n)\underline{\lambda}_k(\sigma^n(\omega), m).\end{aligned}$$

This establishes inequality (5.8). A similar argument shows that (5.9) holds. □

Lemma 5.1.6. *Given $r > 0$, there exists $k \in \mathbb{N}$ such that* $\operatorname{diam} R_{i_1 \cdots i_k} < r$ *for all* $(i_1 i_2 \cdots) \in \Sigma_A^+$.

Proof of the lemma. For each $\omega \in \Sigma_A^+$, let $k(\omega)$ be the least positive integer such that

$$\operatorname{diam} R_{i_1(\omega) \cdots i_{k(\omega)}(\omega)} < r.$$

The cylinder sets $C_{i_1(\omega) \cdots i_{k(\omega)}(\omega)}$ form an open cover of the compact set Σ_A^+. Hence, there exists a finite cover $C^1, \ldots, C^l$ of Σ_A^+, and the sets $\chi(C^1), \ldots, \chi(C^l)$ form a finite cover of X. Moreover, there exist points $\omega_1, \ldots, \omega_l \in \Sigma_A^+$ such that

$$\chi(C^j) = R_{i_1(\omega_j) \cdots i_{k(\omega_j)}(\omega_j)}$$

for $j = 1, \ldots, l$. Let

$$k = \max\{k(\omega_j) : j = 1, \ldots, l\}.$$

Then, for each $(i_1 i_2 \cdots) \in \Sigma_A^+$, there exists $j \in \{1, \ldots, l\}$ such that $R_{i_1 \cdots i_k} \subset \chi(C^j)$. Therefore, $\operatorname{diam} R_{i_1 \cdots i_k} < r$. □

Since X is compact, there exists $r_0 > 0$ such that $h|B(x, r_0)$ is injective for every $x \in X$. Take a positive integer k as in Lemma 5.1.6, corresponding to $r = r_0$. Fix $\omega = (i_1 i_2 \cdots) \in \Sigma_A^+$, and take $x, y \in R_{i_1 \cdots i_k}$ such that $d(x, y) = r' < r_0$. For each $r \in (r', r_0)$, we have $d(h(x), h(y)) < ar$, because g is expanding. Letting $r \to r'$, we obtain

$$d(h(x), h(y)) \leq ad(x, y),$$

and hence, $\underline{\lambda}_{k-1}(\omega, 1) \geq a^{-1}$.

Take now $r \in (0, r')$. Then $y \notin B(x, r)$. Since $h|B(x, r_0)$ is injective, we have $h(y) \notin h(B(x, r))$. Moreover, since h is expanding, we obtain $h(y) \notin B(h(x), br)$. Hence, $d(h(x), h(y)) \geq br$, and letting $r \to r'$, we obtain

$$d(h(x), h(y)) \geq bd(x, y).$$

This shows that $\overline{\lambda}_{k-1}(\omega, 1) \leq b^{-1}$. The proposition follows now readily from Lemma 5.1.5. □

5.1.2 Dimension estimates

We obtain in this section dimension estimates for repellers of continuous expanding maps. The main tool is the nonadditive thermodynamic formalism developed in Chapter 4, and in particular the somewhat more explicit formulas for the topological pressure in the case of symbolic dynamics.

Since the functions defined by (5.7) are constant on each set $R_{i_1\cdots i_n}$, it follows from Proposition 5.1.4 and the second property in Theorem 4.4.1 that there exist unique roots $\underline{s}_k$ and $\overline{s}_k$ respectively of the equations

$$\overline{P}_{\Sigma_A^+}(s\underline{\Phi}_k) = 0 \quad \text{and} \quad P_{\Sigma_A^+}(s\overline{\Phi}_k) = 0.$$

The following result gives dimension estimates for a repeller in terms of these roots.

Theorem 5.1.7 ([5]). *If X is a repeller of a topologically mixing continuous expanding map, then*

$$\sup_{k\in\mathbb{N}} \underline{s}_k \le \dim_H X \le \underline{\dim}_B X \le \overline{\dim}_B X \le \inf_{k\in\mathbb{N}} \overline{s}_k.$$

Proof. Let $\mathcal{U}_n$ be the open cover of Σ_κ^+ formed by the n-cylinder sets. Clearly, $\operatorname{diam} \mathcal{U}_n \to 0$ when $n \to \infty$. We first prove an auxiliary statement.

Lemma 5.1.8. *If $P_Q(s\Phi) = 0$ for $s \in \mathbb{R}$, where $\varphi_n(\omega) = \log \lambda_{i_1\cdots i_n}$ for some numbers $\lambda_{i_1\cdots i_n} > 0$, then for each $\delta > 0$ and all sufficiently large $m \in \mathbb{N}$, there exists a cover of Q by disjoint $(n_{mj} + m)$-cylinder sets $C_{I_{mj}} \subset \Sigma_\kappa^+$, for $j = 1, \dots, N_m < \infty$, such that*

$$\sum_{j=1}^{N_m} \exp\left(-\delta n_{mj}\right) \lambda_{I_{mj}}^s \le e^{\delta m}.$$

Proof of the lemma. We have

$$P_Q(s\Phi) = \lim_{l\to\infty} P_Q(s\Phi, \mathcal{U}_l) = 0.$$

Given $\delta > 0$, take $l \in \mathbb{N}$ such that $|P_Q(s\Phi, \mathcal{U}_l)| < \delta/2$. Then $M_Q(\delta, s\Phi, \mathcal{U}_l) = 0$, and there exists $m_0 \in \mathbb{N}$ such that for each $m \ge m_0$, we have

$$\sum_{U\in\Gamma_m} \exp\left(-\delta m(U) + s \sup_{\Sigma_\kappa^+(U)} \varphi_{m(U)}\right) < 1$$

for some finite collection $\Gamma_m \subset \bigcup_{n\ge m} \mathcal{W}_n(\mathcal{U}_l)$ covering Q. We note that since the set Q is compact, such a cover always exists.

We write

$$\left\{\Sigma_\kappa^+(U) : U \in \Gamma_m\right\} = \left\{C_{J_{mj}} : j = 1, \dots, N_m\right\},$$

where J_{mj} is a Q-admissible finite sequence for each $j = 1, \ldots, N_m < \infty$. If the sequence J_{mj} has $n_{mj} + m + l - 1$ components, we denote by I_{mj} its first $n_{mj} + m$ components. Then

$$\sum_{j=1}^{N_m} \exp(-\delta n_{mj}) \lambda_{I_{mj}}^s \le e^{\delta m}. \tag{5.10}$$

Moreover, we can redefine I_{mj} and N_m so that $C_{I_{mj}} \cap C_{I_{mk}} = \varnothing$ whenever $j \neq k$. This does not affect (5.10), and hence, the desired result follows. □

Now let $\mathcal{U}$ be a family of subsets of X. We also consider a Markov partition of X. Since h is topologically mixing, for the corresponding transition matrix A there exists $l \in \mathbb{N}$ such that A^l has only positive entries. For each $(i_1 i_2 \cdots) \in \Sigma_A^+$ and $n \in \mathbb{N}$, we define

$$\overline{\mathcal{U}}(\omega, n) = \left\{ (h^n | R_{i_1 \cdots i_n})^{-1} U : U \in \mathcal{U} \right\}$$

and

$$\underline{\mathcal{U}}(\omega, n) = \left\{ h^{n+l-1}(U) : U \in \mathcal{U}, U \cap R_{i_1 \cdots i_n} \neq \varnothing \right\}.$$

Lemma 5.1.9. *Let $\mathcal{U}$ be a cover of X. For each $\omega = (i_1 i_2 \cdots) \in \Sigma_A^+$ and $n \in \mathbb{N}$:*

1. *$\overline{\mathcal{U}}(\omega, n)$ is a cover of $R_{i_1 \cdots i_n}$;*
2. *$\underline{\mathcal{U}}(\omega, n)$ is a cover of X.*

Proof of the lemma. The first property follows immediately from the inclusion $h^n(R_{i_1 \cdots i_n}) \subset X$. For the second property, we observe that since A^l has only positive entries, we have $\sigma^{n+l-1}(C_{i_1 \cdots i_n}) = \Sigma_A^+$ for each cylinder set $C_{i_1 \cdots i_n}$. Hence,

$$h^{n+l-1}(R_{i_1 \cdots i_n}) = (h^{n+l-1} \circ \chi)(C_{i_1 \cdots i_n}) = \chi(\Sigma_A^+) = X,$$

and $\underline{\mathcal{U}}(\omega, n)$ is a cover of X. □

The following result gives an upper bound for the upper box dimension.

Lemma 5.1.10. *We have $\overline{\dim}_B X \le \overline{s}_k$ for all sufficiently large $k \in \mathbb{N}$.*

Proof of the lemma. Let us take $k \in \mathbb{N}$ sufficiently large as in Proposition 5.1.4. Given $\delta > 0$, it follows from Lemma 5.1.8 that for each sufficiently large $m \in \mathbb{N}$, there exists a cover of Σ_A^+ by $(n_{mj} + m)$-cylinder set $C_{I_{mj}} \subset \Sigma_\kappa^+$, with $j = 1, \ldots, N_m < \infty$, such that

$$\sum_{j=1}^{N_m} \exp(-\delta n_{mj}) \overline{\lambda}_{mjk}^{\overline{s}_k} \le e^{\delta_m}, \tag{5.11}$$

where $\overline{\lambda}_{mjk} = \overline{\lambda}_k(\omega, n_{mj} + m)$ when the first $n_{mj} + m$ components of ω are I_{mj}.

Now let $N_\delta(Z)$ be the least number of sets of diameter at most δ needed to cover Z. We assume that $N_\delta(X) \le a(\delta)$ for each $\delta \in (0, \delta_0)$, for some function a

and some $\delta_0 > 0$. By the first property in Lemma 5.1.9, if $\mathcal{U}$ is a cover of X, then $\overline{\mathcal{U}}(\omega, n_{mj} + m)$ is a cover of $R_{I_{mj}}$, with

$$\operatorname{diam} \overline{\mathcal{U}}(\omega, n_{mj} + m) \le \overline{\lambda}_{mjk} \operatorname{diam} \mathcal{U}.$$

Therefore, $N_{\delta \overline{\lambda}_{mjk}}(R_{I_{mj}}) \le a(\delta)$, and hence,

$$N_\delta(R_{I_{mj}}) \le a(\delta/\overline{\lambda}_{mjk}) \quad \text{for every} \quad \delta < \delta_0 \lambda_{mk},$$

where

$$\lambda_{mk} = \min\left\{\overline{\lambda}_{mjk} : j = 1, \ldots, N_m\right\}.$$

We have $\lambda_{mk} < 1$, and since $N_m < \infty$, we also have $\lambda_{mk} > 0$. Therefore, for each $\delta < \delta_0 \lambda_{mk}$ we obtain

$$N_\delta(X) \le \sum_{j=1}^{N_m} N_\delta(R_{I_{mj}}) \le \sum_{j=1}^{N_m} a(\delta/\overline{\lambda}_{mjk}). \tag{5.12}$$

Let us take $s > 0$ such that $\overline{\dim}_B X < s$. Then there exists $\delta_0 > 0$ such that $N_\delta(X) \le \delta^{-s}$ for all $\delta \in (0, \delta_0)$. When $a(\delta) = \delta^{-s}$, it follows from (5.12) that

$$N_\delta(X) \le \delta^{-s} c_{mk}(s) := \delta^{-s} \sum_{j=1} \ N_m \overline{\lambda}_{mjk}^s,$$

for every $\delta < \delta_0 \lambda_{mk}$. We can now restart the process using the function $a(\delta) = \delta^{-s} c_{mk}(s)$, and it follows by induction that

$$N_\delta(X) \le \delta^{-s} c_{mk}(s)^l \quad \text{for every} \quad \delta < \delta_0 \lambda_{mk}^l.$$

Therefore,

$$\begin{aligned}
\frac{\log N_\delta(X)}{-\log \delta} &\le \frac{\log(\delta^{-s} c_{mk}(s)^l)}{-\log \delta} \\
&= s + \frac{l \log c_{mk}(s)}{-\log \delta} \le s + \frac{l \log c_{mk}(s)}{-\log(\delta_0 \lambda_{mk}^l)},
\end{aligned}$$

and letting $l \to \infty$, we obtain

$$\overline{\dim}_B X \le s + \limsup_{l \to \infty} \frac{l \log c_{mk}(s)}{-\log(\delta_0 \lambda_{mk}^l)} = s + \frac{\log c_{mk}(s)}{-\log \lambda_{mk}}.$$

Since $N_m < \infty$, the function $s \mapsto c_{mk}(s)$ is continuous for each k. Thus, we can let $s \searrow \overline{\dim}_B X$ to obtain

$$\log c_{mk}(\overline{\dim}_B X)/(-\log \lambda_{mk}) \ge 0.$$

Since $\lambda_{mk} < 1$ for all sufficiently large k, we conclude that $c_{mk}(\overline{\dim}_B X) \ge 1$.

Now assume that $\overline{\dim}_B X > \overline{s}_k$, and take $\delta > 0$ such that

$$d := -(\overline{\dim}_B X - \overline{s}_k)\log b + \delta < 0.$$

By (5.11), we obtain

$$\begin{aligned}
1 &\le \sum_{j=1}^{N_m} \overline{\lambda}_{I_{mjk}}^{\overline{\dim}_B X} \\
&\le \sum_{j=1}^{N_m} \Big(\exp(-\delta n_{mj}) \overline{\lambda}_{I_{mjk}}^{\overline{s}_k} e^{-\delta m} \Big) \times \Big(\overline{\lambda}_{I_{mjk}}^{\overline{\dim}_B X - \overline{s}_k} \exp(\delta(n_{mj}+m)) \Big) \\
&\le \sum_{j=1}^{N_m} \Big(\exp(-\delta n_{mj}) \overline{\lambda}_{I_{mjk}}^{\overline{s}_k} e^{-\delta m} \Big) \exp[d(n_{mj}+m)] \\
&\le \exp(-dm) < 1.
\end{aligned}$$

This contradiction shows that $\overline{\dim}_B X \le \overline{s}_k$. □

Now we establish a lower bound for the Hausdorff dimension.

Lemma 5.1.11. *We have* $\dim_H X \ge \underline{s}_k$ *for all sufficiently large* $k \in \mathbb{N}$.

Proof of the lemma. Let

$$q = \max_{1 \le i \le \kappa} \operatorname{card}\big\{1 \le j \le \kappa : R_i \cap R_j \ne \varnothing\big\}.$$

Clearly, if the Markov partition has sufficiently small diameter, then $q < \kappa$. Moreover, for each $n \in \mathbb{N}$ the number of sets $R_{j_1 \cdots j_n}$ intersecting a given set $R_{i_1 \cdots i_n}$ is at most q, and there exists $\delta_n > 0$ such that each ball of radius δ_n intersects at most q sets $R_{i_1 \cdots i_n}$.

Given a sufficiently large $k \ge l-1$ as in Proposition 5.1.4, let us assume on the contrary that $\dim_H X < \underline{s}_k$. Let also s be a positive number such that $\dim_H X < s < \underline{s}_k$. Then the s-dimensional Hausdorff measure of X (see (1.6)) is $m_H(X,s) = 0$, and since X is compact, there exists a finite open cover $\mathcal{U}$ of X such that

$$\sum_{U \in \mathcal{U}} (\operatorname{diam} U)^s < \delta_{n+k}^s.$$

In particular, for each $U \in \mathcal{U}$ we have $\operatorname{diam} U < \delta_{n+k}$, and the set U intersects at most a number q of the sets $R_{i_1 \cdots i_{n+k}}$. By the second property in Lemma 5.1.9, for each $\omega = (i_1 i_2 \cdots) \in \Sigma_A^+$ and $n \in \mathbb{N}$, the family $\underline{\mathcal{U}}(\omega, n)$ is a cover of X. On the other hand, by Lemma 5.1.5 we have

$$\begin{aligned}
\operatorname{diam} h^{n+l-1}(U) &\le \underline{\lambda}_k(\omega, n+l-1)^{-1} \operatorname{diam} U \\
&\le \big(\underline{\lambda}_k(\omega, n)\underline{\lambda}_k(\sigma^n(\omega), l-1)\big)^{-1} \operatorname{diam} U \\
&\le a^{l-1} \underline{\lambda}_k(\omega, n)^{-1} \operatorname{diam} U
\end{aligned}$$

for each $U \in \mathfrak{U}$ such that $U \cap R_{i_1 \cdots i_n} \neq \varnothing$. Hence,

$$\sum_{U \in \underline{\mathfrak{U}}(\omega,n)} (\operatorname{diam} U)^s \leq a^{(l-1)s} \underline{\lambda}_k(\omega,n)^{-s} \sum_{U \in \mathfrak{U}, U \cap R_{i_1 \cdots i_n} \neq \varnothing} (\operatorname{diam} U)^s.$$

Let us assume that

$$\sum_{U \in \underline{\mathfrak{U}}(\omega,n)} (\operatorname{diam} U)^s \geq \delta_{n+k}^s$$

for each $\omega = (i_1 i_2 \cdots) \in \Sigma_A^+$ and $n \in \mathbb{N}$. Then, for each $n \in \mathbb{N}$, we obtain

$$\begin{aligned}
q\delta_{n+k}^s &> q \sum_{U \in \mathfrak{U}} (\operatorname{diam} U)^s \\
&\geq \sum_{(i_1 \cdots i_n) \in Q_n} \sum_{U \in \mathfrak{U}, U \cap R_{i_1 \cdots i_n} \neq \varnothing} (\operatorname{diam} U)^s \\
&\geq a^{-(l-1)s} \sum_{(i_1 \cdots i_{k_n}) \in Q_{k_n}} \left(\underline{\lambda}_k(\omega,n)^s \sum_{U \in \underline{\mathfrak{U}}(\omega,n)} (\operatorname{diam} U)^s \right) \\
&\geq a^{-(l-1)s} \delta_{n+k}^s \sum_{(i_1 \cdots i_{k_n}) \in Q_{k_n}} \underline{\lambda}_k(\omega,n)^s.
\end{aligned} \tag{5.13}$$

Now we observe that by Theorem 4.5.1, there exists a sequence of positive integers $k_n \nearrow +\infty$ such that

$$\lim_{n \to \infty} \frac{1}{k_n} \sum_{(i_1 \cdots i_{k_n}) \in Q_{k_n}} \underline{\lambda}_k(\omega,k_n)^{\underline{s}_k} = 0. \tag{5.14}$$

Since $(\underline{s}_k - s) \log b > 0$, one can choose ε such that

$$(\underline{s}_k - s) \log b - \varepsilon > 0, \tag{5.15}$$

and

$$\sum_{(i_1 \cdots i_{k_n}) \in Q_{k_n}} \underline{\lambda}_k(\omega,k_n)^{\underline{s}_k} \geq e^{-\varepsilon k_n}$$

for all sufficiently large n. By (5.13), (5.14), the inequality $s < \underline{s}_k$, and Proposition 5.1.4, we thus obtain

$$\begin{aligned}
qa^{(l-1)s} &> \sum_{(i_1 \cdots i_{k_n}) \in Q_{k_n}} \underline{\lambda}_k(\omega,k_n)^s \\
&\geq b^{(\underline{s}_k - s)k_n} \sum_{(i_1 \cdots i_{k_n}) \in Q_{k_n}} \underline{\lambda}_k(\omega,k_n)^{\underline{s}_k} \\
&\geq \exp\left[(\underline{s}_k - s)k_n \log b - \varepsilon k_n\right].
\end{aligned}$$

But in view of (5.15), this inequality cannot hold for any sufficiently large n. This contradiction shows that for each sufficiently large n, and any finite cover $\mathcal{U}$ of X satisfying

$$\sum_{U\in\mathcal{U}}(\operatorname{diam} U)^s < \delta^s_{k_n+k},$$

we must have

$$\sum_{U\in\underline{\mathcal{U}}(\omega',k_n)}(\operatorname{diam} U)^s < \delta^s_{k_n+k} \quad \text{for some} \quad \omega' \in \Sigma_A^+.$$

We can now restart the process using $\mathcal{U}^1 = \underline{\mathcal{U}}(\omega', k_n)$ instead of $\mathcal{U}$. In this way, we find inductively a sequence of finite covers $\mathcal{U}^l$ of X, for each $l \in \mathbb{N}$. Since each set $U \in \mathcal{U}^l$ intersects at most a number q of the sets $R_{i_1\cdots i_n}$, for some $q < \kappa$, we have $\operatorname{card}\mathcal{U}^{l+1} < \operatorname{card}\mathcal{U}^l$. This implies that $\operatorname{card}\mathcal{U}^{l(n)} = 1$ for some $l(n) \in \mathbb{N}$. Writing $\mathcal{U}^{l(n)} = \{U_n\}$, we obtain

$$0 < \operatorname{diam} X \le \operatorname{diam} U_n < \delta_{k_n+k} \to 0 \quad \text{when} \quad n \to \infty,$$

which is impossible. This contradiction shows that $\dim_H X \ge \underline{s}_k$. □

The desired estimates in Theorem 5.1.7 follow now immediately from Lemmas 5.1.10 and 5.1.11. □

We recall that the sequence $k \mapsto \underline{\lambda}_k(\omega, n)$ is nondecreasing and that the sequence $k \mapsto \overline{\lambda}_k(\omega, n)$ is nonincreasing, for each given $\omega \in \Sigma_A^+$ and $n \in \mathbb{N}$. By the first property in Theorem 4.2.2, the sequences $k \mapsto \underline{s}_k$ and $k \mapsto \overline{s}_k$ are respectively nondecreasing and nonincreasing. Moreover, by the last property in Theorem 4.4.1, since the topological entropy $h(\sigma|\Sigma_A^+)$ is positive, we have $\underline{s}_k > 0$ for all sufficiently large k.

The dimension of repellers has a strong homogeneity property.

Theorem 5.1.12. *If X is a repeller of a topologically transitive continuous expanding map, then*

$$\begin{aligned}\dim_H(X\cap U) &= \dim_H X,\\ \underline{\dim}_B(X\cap U) &= \underline{\dim}_B X,\\ \overline{\dim}_B(X\cap U) &= \overline{\dim}_B X\end{aligned}$$

for each nonempty open set U in X.

Proof. Since h is topologically transitive, the matrix A is irreducible, that is, for each i and j there exists $n \in \mathbb{N}$ such that the (i,j)-entry of A^n is positive. This guarantees that for each $j = 1, \ldots, \kappa$, and each open set U such that $X \cap U \ne \varnothing$, there exist $(i_1 i_2 \cdots) \in \Sigma_A^+$ and $n \in \mathbb{N}$, such that $i_{n+1} = j$ and $R_{i_1\cdots i_n} \subset X \cap U$.

Hence,

$$\begin{aligned}\dim_H R_j &\le \dim_H h^n(R_{i_1\cdots i_n})\\ &= \max_{(i_{n+1}\cdots i_{n+k})} \dim_H h^n(R_{i_1\cdots i_{n+k}})\\ &\le \max_{(i_{n+1}\cdots i_{n+k})} \dim_H R_{i_1\cdots i_{n+k}} \qquad (5.16)\\ &= \dim_H R_{i_1\cdots i_n}\\ &\le \dim_H(X\cap U) \le \dim_H X,\end{aligned}$$

because the map h^n is Lipschitz on each set $R_{i_1\cdots i_{n+k}}$. Since

$$\dim_H X = \max\bigl\{\dim_H R_j : j = 1,\ldots,\kappa\bigr\},$$

it follows from (5.16) that $\dim_H(X\cap U) = \dim_H X$. Similar arguments apply to the lower and upper box dimensions. □

5.1.3 Conformal maps

We consider in this section the particular class of repellers of asymptotically conformal expanding maps. For this class, the lower and upper dimension estimates in the former section coincide, and thus, inequalities such as those in Theorem 5.1.7 become equalities.

We first recall the notion of asymptotically conformal expanding map (with respect to a given Markov partition).

Definition 5.1.13. The expanding map $h\colon X \to X$ is said to be *asymptotically conformal* if there exists $k \in \mathbb{N}$ such that

$$\frac{1}{n}\log\frac{\overline{\lambda}_k(\omega,n)}{\underline{\lambda}_k(\omega,n)} \to 0 \text{ uniformly on } \Sigma_A^+ \text{ when } n\to\infty,$$

where the numbers $\underline{\lambda}_k(\omega,n)$ and $\overline{\lambda}_k(\omega,n)$ are the lower and upper ratio coefficients in (5.5) and (5.6) with respect to a given Markov partition.

For repellers of asymptotically conformal expanding maps, we can formulate a much stronger statement than that in Theorem 5.1.7.

Theorem 5.1.14 ([5]). *Let X be a repeller of a topologically mixing and asymptotically conformal expanding map. For each nonempty open set U in X, and all sufficiently large $k \in \mathbb{N}$, we have*

$$\dim_H(X\cap U) = \underline{\dim}_B(X\cap U) = \overline{\dim}_B(X\cap U) = \underline{s}_k = \overline{s}_k = s,$$

where s is the unique root of the equation

$$\lim_{n\to\infty}\frac{1}{n}\log\sum_{(i_1\cdots i_n)\in S_n}\bigl(\operatorname{diam} R_{i_1\cdots i_n}\bigr)^s = 0,$$

and where S_n is the set of all Σ_A^+-admissible sequences of length n.

Proof. We start with an auxiliary result which establishes the relation between the ratio coefficients and the diameters of the sets $R_{i_1\cdots i_n}$ in (5.4).

Lemma 5.1.15. *For each $k \in \mathbb{N}$, there exist constants $C_1, C_2 > 0$ such that*

$$C_1 \underline{\lambda}_k(\omega, n) \le \operatorname{diam} R_{i_1\cdots i_{n+k}} \le C_2 \overline{\lambda}_k(\omega, n)$$

for every $\omega = (i_1 i_2 \cdots) \in Q$ and $n \in \mathbb{N}$.

Proof of the lemma. It follows from the definitions that

$$\underline{\lambda}_k(\omega, n) \le \operatorname{diam} R_{i_1\cdots i_{n+k}} / \operatorname{diam} h^n(R_{i_1\cdots i_{n+k}}) \le \overline{\lambda}_k(\omega, n).$$

Hence, for each given $k \in \mathbb{N}$ we have

$$C_1 \underline{\lambda}_k(\omega, n) \le \operatorname{diam} R_{i_1\cdots i_{n+k}} \le C_2 \overline{\lambda}_k(\omega, n),$$

where

$$C_1 = \inf \left\{ \operatorname{diam} h^n(R_{i_1\cdots i_{n+k}}) : (i_1 i_2 \cdots) \in Q \text{ and } n \in \mathbb{N} \right\}$$

and $C_2 = \operatorname{diam} X > 0$. Moreover, since $h^n(R_{i_1\cdots i_{n+k}}) = h(R_{i_n\cdots i_{n+k}})$, we obtain

$$C_1 = \min \left\{ \operatorname{diam} h(R_{j_1\cdots j_{k+1}}) : (j_1 \cdots j_{k+1}) \in Q_{k+1} \right\} > 0.$$

This yields the desired result. □

Since h is asymptotically conformal, given $\varepsilon > 0$, we have

$$\underline{\lambda}_k(\omega, n) \le \overline{\lambda}_k(\omega, n) \le e^{\varepsilon n} \underline{\lambda}_k(\omega, n)$$

for every $\omega \in \Sigma_A^+$ and all sufficiently large $n, k \in \mathbb{N}$. By the first property in Theorem 4.2.2, we obtain

$$\underline{P}_{\Sigma_A^+}(s\underline{\Phi}_k) \le \underline{P}_{\Sigma_A^+}(s\overline{\Phi}_k) \le \underline{P}_{\Sigma_A^+}(s\underline{\Phi}_k) + \varepsilon$$

and

$$\overline{P}_{\Sigma_A^+}(s\underline{\Phi}_k) \le \overline{P}_{\Sigma_A^+}(s\overline{\Phi}_k) \le \overline{P}_{\Sigma_A^+}(s\underline{\Phi}_k) + \varepsilon,$$

for each $s \in \mathbb{R}$. Since ε is arbitrary, we conclude that

$$\underline{P}_{\Sigma_A^+}(s\underline{\Phi}_k) = \underline{P}_{\Sigma_A^+}(s\overline{\Phi}_k), \tag{5.17}$$

and hence,

$$\begin{aligned} \overline{P}_{\Sigma_A^+}(s\underline{\Phi}_k) = \overline{P}_{\Sigma_A^+}(s\overline{\Phi}_k) &\ge \underline{P}_{\Sigma_A^+}(s\underline{\Phi}_k) \\ = \underline{P}_{\Sigma_A^+}(s\overline{\Phi}_k) &\ge P_{\Sigma_A^+}(s\overline{\Phi}_k) \end{aligned} \tag{5.18}$$

for each $s \in \mathbb{R}$. It thus follows from Theorem 5.1.7 and (5.18) that $\underline{s}_k = \overline{s}_k$ for all sufficiently large $k \in \mathbb{N}$. On the other hand, by Lemma 5.1.5 and Theorem 4.2.6, we have $\overline{P}_{\Sigma_A^+}(s\overline{\Phi}_k) = \underline{P}_{\Sigma_A^+}(s\overline{\Phi}_k)$. By Lemma 5.1.15 and Theorem 4.5.1, we conclude that

$$\begin{aligned}\underline{P}_{\Sigma_A^+}(s\underline{\Phi}_k) &= \liminf_{n\to\infty} \frac{1}{n} \log \sum_{(i_1\cdots i_n)\in S_n} \underline{\lambda}_k(\omega,n)^s \\ &\le \liminf_{n\to\infty} \frac{1}{n} \log \sum_{(i_1\cdots i_{n+k})\in S_{n+k}} \big(\operatorname{diam} R_{i_1\cdots i_{n+k}}\big)^s \\ &\le \limsup_{n\to\infty} \frac{1}{n} \log \sum_{(i_1\cdots i_{n+k})\in S_{n+k}} \big(\operatorname{diam} R_{i_1\cdots i_{n+k}}\big)^s \\ &\le \limsup_{n\to\infty} \frac{1}{n} \log \sum_{(i_1\cdots i_{n+k})\in S_{n+k}} \overline{\lambda}_k(\omega,n)^s \\ &\le \limsup_{n\to\infty} \frac{1}{n} \log \left(\kappa^k \sum_{(i_1\cdots i_n)\in S_n} \overline{\lambda}_k(\omega,n)^s\right) \\ &= \overline{P}_{\Sigma_A^+}(s\overline{\Phi}_k) = \underline{P}_{\Sigma_A^+}(s\overline{\Phi}_k) = \underline{P}_{\Sigma_A^+}(s\underline{\Phi}_k),\end{aligned} \tag{5.19}$$

using (5.17) in the last identity. The desired result follows now readily from Theorems 5.1.7 and 5.1.12 together with (5.19). □

5.2 Repellers of smooth expanding maps

We consider in this section the particular case of smooth expanding maps, and we obtain dimension estimates for their repellers. Again, the main tool is the nonadditive thermodynamic formalism developed in Chapter 4. Some of the results are obtained as a consequence of corresponding results for continuous expanding maps. We emphasize that in general the smooth maps that we consider are only of class C^1, and thus, the bounded distortion property may not hold.

5.2.1 Basic notions

We first introduce the notion of repeller of a smooth expanding map. Roughly speaking, a smooth map is expanding if its differential expands every nonzero vector, eventually up to a multiplicative factor.

Let $f\colon M \to M$ be a differentiable transformation of a smooth manifold, and let $J \subset M$ be a compact f-invariant set (which means that $f^{-1}J = J$).

Definition 5.2.1. We say that J is a *repeller* of f and that f is *expanding* on J if there exist constants $c > 0$ and $\beta > 1$ such that

$$\|d_x f^n v\| \ge c\beta^n \|v\| \tag{5.20}$$

for every $n \in \mathbb{N}$, $x \in J$, and $v \in T_x M$.

It follows from (5.20) that the linear transformation $d_x f^n$ is invertible for every $x \in J$ and $n \in \mathbb{N}$. In particular, an expanding map f is a local diffeomorphism at every point, that is, each $x \in J$ has an open neighborhood U_x such that $f|U_x\colon U_x \to f(U_x)$ is invertible and has differentiable inverse.

We describe two examples of expanding maps and repellers.

Example 5.2.2 (Rational maps). Consider a rational map $f\colon S \to S$ of degree at least 2 on the Riemann sphere S. An n-periodic point x (which means that $f^n(x) = x$) is said to be *repelling* if $|(f^n)'(x)| > 1$. The *Julia set* $J \subset S$ of f is the closure of the set of repelling periodic points of f. In particular, J is compact, nonempty, and f-invariant. For example, the map $z \mapsto z^2 + c$ is expanding on its Julia set provided that $|c| < 1/4$ (see for example [60] for details), and in this case J is a repeller of f.

Example 5.2.3 (One-dimensional Markov maps). Consider disjoint closed intervals $\Delta_1, \ldots, \Delta_\kappa \subset [0,1]$. The map $f\colon [0,1] \to [0,1]$ is called a *Markov map* if:

1. for each $j = 1, \ldots, \kappa$ there is a set of indices $I(j)$ with $f(\Delta_j) = \bigcup_{i\in I(j)} \Delta_i$;
2. at every point $x \in \Delta := \bigcup_{i=1}^{\kappa} \operatorname{int} \Delta_i$ the derivative of f exists and satisfies $|f'(x)| \geq \gamma$ for some fixed $\gamma > 0$;
3. there exist $\lambda > 1$ and $n \in \mathbb{N}$ such that $|(f^n)'(x)| \geq \lambda$ whenever $f^k(x) \in \Delta$ for $k = 0, \ldots, n$.

A repeller of a Markov map f is given by

$$J = \bigcap_{k=1}^{\infty} \overline{f^{-k}\Delta}.$$

5.2.2 Dimension estimates

We present in this section dimension estimates for repellers of smooth expanding maps that need not be conformal. The main aim is to present the best possible estimates using the least information about the dynamics. When more geometric information is available one can often obtain sharper estimates for the dimension or even compute its value (see Section 5.3.1 for a related discussion). However, this often requires a more elaborate approach.

Given a repeller J of a differentiable transformation g, we consider the functions $\underline{\varphi}, \overline{\varphi}\colon J \to \mathbb{R}$ defined by

$$\underline{\varphi}(x) = -\log\|d_x g\| \quad \text{and} \quad \overline{\varphi}(x) = \log\|(d_x g)^{-1}\|. \tag{5.21}$$

Since g is expanding on J, by Theorem 4.4.1 the functions

$$s \mapsto P_J(s\underline{\varphi}) \quad \text{and} \quad s \mapsto P_J(s\overline{\varphi})$$

are Lipschitz and strictly decreasing, and thus there exist unique roots $\underline{t}$ and $\overline{t}$ respectively of the equations

$$P_J(\underline{t\varphi}) = 0 \quad \text{and} \quad P_J(\overline{t}\,\overline{\varphi}) = 0.$$

The relation between the functions $\underline{\varphi}$ and $\overline{\varphi}$, and those in (5.7) is the following. Given $\varepsilon > 0$, there exist a Markov partition of J with sufficiently small diameter and $k \in \mathbb{N}$, such that

$$-n\varepsilon + \sum_{j=0}^{n-1} \underline{\varphi} \circ g^j \leq \underline{\varphi}_{k,n} \leq \overline{\varphi}_{k,n} \leq n\varepsilon + \sum_{j=0}^{n-1} \overline{\varphi} \circ g^j$$

for every $n \in \mathbb{N}$ (this follows from a slight modification of the proof of Proposition 5.2.11). These inequalities readily imply that

$$\underline{t} \leq \underline{s}_k \leq \overline{s}_k \leq \overline{t}$$

for any sufficiently large $k \in \mathbb{N}$. The following result is thus an immediate consequence of Theorem 5.1.7.

Theorem 5.2.4. *Let J be a repeller of a C^1 expanding map g which is topologically mixing on J. Then*

$$\underline{t} \leq \dim_H J \leq \underline{\dim}_B J \leq \overline{\dim}_B J \leq \overline{t}. \tag{5.22}$$

The upper bound in the right-hand side of (6.2) was obtained by Gelfert in [75] in the more general case of volume expanding maps. In a related direction, Shafikov and Wolf showed in [176] that for a repeller J of a C^2 map $g\colon M \to M$ we have

$$\overline{\dim}_B J \leq \dim M + \frac{P_J(-\log|\det dg|)}{\lambda},$$

where

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \log \max_{x\in J} \|d_x g^n\|.$$

Now we obtain dimension estimates for a repeller J using certain sequences of functions and the nonadditive thermodynamic formalism. Example 5.2.9 illustrates that these estimates may be sharper than those in Theorem 5.2.4.

Given a repeller J of a differentiable map $g\colon M \to M$, we consider again a Markov partition $R_1, \ldots, R_\kappa$ of J (see Definition 5.1.2, where the interior of each set R_i is computed with respect to the induced topology on J). We recall that any repeller has Markov partitions with arbitrarily small diameter (see [167]). We also consider the associated topological Markov chain $\sigma|\Sigma_A^+$ with transition matrix $A = (a_{ij})$ given by

$$a_{ij} = \begin{cases} 1 & \text{if } g(\operatorname{int} R_i) \cap \operatorname{int} R_j \neq \varnothing, \\ 0 & \text{if } g(\operatorname{int} R_i) \cap \operatorname{int} R_j = \varnothing. \end{cases}$$

For each $i = 1, \ldots, \kappa$, let $\hat{R}_i$ be a sufficiently small open neighborhood of R_i such that $\hat{R}_i \cap \hat{R}_j = \varnothing$ whenever $R_i \cap R_j = \varnothing$. We also define sets $\hat{R}_{i_1 \cdots i_n}$ in a similar manner to that in (5.4) with R_{i_k} replaced by $\hat{R}_{i_k}$ for each k, that is,

$$\hat{R}_{i_1 \cdots i_n} = \bigcap_{k=0}^{n-1} g^{-k} \hat{R}_{i_{k+1}}. \tag{5.23}$$

Now we consider the sequences of functions $\underline{\varphi}_n$ and $\overline{\varphi}_n$ in Σ_A^+ given by

$$\underline{\varphi}_n(x) = -\log \max_{x \in \hat{R}_{i_1 \cdots i_n}} \|d_x g^n\| \tag{5.24}$$

and

$$\overline{\varphi}_n(x) = \log \max_{x \in \hat{R}_{i_1 \cdots i_n}} \|(d_x g^n)^{-1}\| \tag{5.25}$$

for each $\omega = (i_1 i_2 \cdots) \in \Sigma_A^+$, and we denote them respectively by $\underline{\Phi}$ and $\overline{\Phi}$. We note that there exist constants $C > 0$ and $a > 1$ such that

$$Ca^n \leq \|(d_x g^n)^{-1}\|^{-1} \leq \|d_x g^n\| \leq K^n \tag{5.26}$$

for every $x \in J$ and $n \in \mathbb{N}$, where

$$K = \max \bigl\{ \|d_x g\| : x \in J \bigr\}.$$

Clearly, the sequences $\underline{\Phi}$ and $\overline{\Phi}$ have tempered variation (see (4.2)). By (5.26) and the second property in Theorem 4.4.1, there exist unique roots $\underline{s}$ and $\overline{s}$ respectively of the equations

$$\overline{P}_{\Sigma_A^+}(s\underline{\Phi}) = 0 \quad \text{and} \quad P_{\Sigma_A^+}(s\overline{\Phi}) = 0.$$

An application of the mean value theorem to g^n and to its local inverses (which exist at every point because g is expanding) shows that given $\varepsilon > 0$, and provided that all the sets $\hat{R}_i$ have sufficiently small diameter, for each $x = \chi(\omega) \in J$ and $n, k \in \mathbb{N}$ we have

$$\begin{aligned} -n\varepsilon + \sum_{j=0}^{n-1} \varphi(g^j(x)) &\leq \underline{\varphi}_n(\omega) \leq \underline{\varphi}_{k,n}(\omega) \\ &\leq \overline{\varphi}_{k,n}(\omega) \leq \overline{\varphi}_n(\omega) \leq n\varepsilon + \sum_{j=0}^{n-1} \varphi(g^j(x)) \end{aligned} \tag{5.27}$$

(the first and last inequalities follow from a slight modification of the proof of Proposition 5.2.11). Therefore,

$$\underline{t} \leq \underline{s} \leq \underline{s}_k \leq \overline{s}_k \leq \overline{s} \leq \overline{t} \tag{5.28}$$

for all sufficiently large $k \in \mathbb{N}$. The following result is thus an immediate consequence of Theorem 5.1.7.

Theorem 5.2.5. *Let J be a repeller of a C^1 expanding map g which is topologically mixing on J. Then*

$$\underline{s} \leq \dim_H J \leq \underline{\dim}_B J \leq \overline{\dim}_B J \leq \overline{s}.$$

Example 5.2.9 illustrates that the estimates in (5.2.5) may be sharper than those in (5.22).

Now we define sequences of functions $\underline{\varphi}_n^*$ and $\overline{\varphi}_n^*$ in J by

$$\underline{\varphi}_n^*(x) = -\log\|d_x g^n\| \quad \text{and} \quad \overline{\varphi}_n^*(x) = \log\|(d_x g^n)^{-1}\|,$$

and we denote them respectively by $\underline{\Phi}^*$ and $\overline{\Phi}^*$. These can be described as pointwise versions of the sequences $\underline{\Phi}$ and $\overline{\Phi}$ in (5.24) and (5.25). If the new sequences $\underline{\Phi}^*$ and $\overline{\Phi}^*$ have tempered variation (see (4.2)), then by (5.26) and the second property in Theorem 4.4.1 there exist unique roots $\underline{s}^*$ and $\overline{s}^*$ respectively of the equations

$$\overline{P}_J(s\underline{\Phi}^*) = 0 \quad \text{and} \quad P_J(s\overline{\Phi}^*) = 0.$$

In order to relate the new sequences $\underline{\Phi}^*$ and $\overline{\Phi}^*$ with the sequences $\underline{\Phi}$ and $\overline{\Phi}$ we recall a notion introduced in [5].

Definition 5.2.6. Given $\alpha \in (0, 1]$, the derivative of g is said to be *α-bunched* if

$$\|(d_x g)^{-1}\|^{1+\alpha}\|d_x g\| < 1 \quad \text{for every} \quad x \in M. \tag{5.29}$$

The following result was established in [5].

Proposition 5.2.7. *If g is a $C^{1+\alpha}$ expanding map with α-bunched derivative, for some $\alpha \in (0, 1]$, then the sequences $\underline{\Phi}^*$ and $\overline{\Phi}^*$ have the tempered variation property in (4.2), and there exists $\varepsilon \geq 0$ such that*

$$-\varepsilon + \underline{\varphi}_n^*(x) \leq \underline{\varphi}_n(\omega) \leq \underline{\varphi}_n^*(x) \quad \textit{and} \quad \overline{\varphi}_n^*(x) \leq \overline{\varphi}_n(\omega) \leq \overline{\varphi}_n^*(x) + \varepsilon \tag{5.30}$$

for every $x = \chi(\omega) \in J$ and all sufficiently large $n \in \mathbb{N}$.

Proof. For simplicity of the proof, we assume that $M = \mathbb{R}^p$ for some $p \in \mathbb{N}$. All the arguments can be generalized in a simple manner to consider the general case.

The following approach is adapted from Falconer [59]. For each $x, y \in M$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}
&\|d_x g^n (d_y g^n)^{-1}\| \\
&= \left\|d_{g(x)} g^{n-1} d_x g (d_y g)^{-1} (d_{g(y)} g^{n-1})^{-1}\right\| \\
&= \left\|d_{g(x)} g^{n-1} (d_{g(y)} g^{n-1})^{-1} [\mathrm{Id} + d_{g(y)} g^{n-1} (d_x g (d_y g)^{-1} - \mathrm{Id})(d_{g(y)} g^{n-1})^{-1}]\right\|,
\end{aligned}$$

and hence,

$$\begin{aligned}
&\frac{\|d_x g^n (d_y g^n)^{-1}\|}{\|d_{g(x)} g^{n-1} (d_{g(y)} g^{n-1})^{-1}\|} \\
&\leq 1 + C_1 \|(d_{g(y)} g^{n-1})^{-1}\| \cdot \|d_{g(y)} g^{n-1}\| \cdot \|d_x g - d_y g\| \\
&\leq 1 + C_2 \|(d_{g(y)} g^{n-1})^{-1}\| \cdot \|d_{g(y)} g^{n-1}\| \cdot \|x - y\|^\alpha,
\end{aligned} \tag{5.31}$$

for some constants $C_1, C_2 > 0$. One can take

$$C_1 = \max\{\|(d_y g)^{-1}\| : y \in M\}.$$

Now we assume that x and y belong to the same set $\hat{R}_{i_1\cdots i_n}$, and we denote by h the local inverse of $g^{n-1}|\hat{R}_{i_1\cdots i_n}$. Then

$$\begin{aligned}\|x-y\| &= \|h(g^{n-1}(x)) - h(g^{n-1}(y))\| \\ &\le \|d_z h\| \cdot \|g^{n-1}(x) - g^{n-1}(y)\|\end{aligned} \tag{5.32}$$

for some point z in the segment between $g^{n-1}(x)$ and $g^{n-1}(y)$. Since the function $x \mapsto \|(d_x g)^{-1}\|$ is continuous, given $\delta > 0$ one can always assume that the diameter of the Markov partition is such that

$$e^{-\delta} \le \|(d_{g^k(x)} g)^{-1}\| / \|(d_{g^k(h(z))} g)^{-1}\| \le e^{\delta}$$

for $k = 0, \ldots, n-2$. Moreover, since the derivative of g is α-bunched, we can choose $\lambda < 1$ such that

$$\|(d_x g)^{-1}\|^{1+\alpha} \|d_x g\| < \lambda$$

for every $x \in M$. Since $d_z h = (d_{h(z)} g^{n-1})^{-1}$, we obtain

$$\begin{aligned}&\|d_z h\|^{\alpha} \cdot \|(d_{g(y)} g^{n-1})^{-1}\| \cdot \|d_{g(y)} g^{n-1}\| \\ &\le e^{\alpha\delta(n-1)} \prod_{k=1}^{n-1} \|(d_{g^k(y)} g)^{-1}\|^{1+\alpha} \|d_{g^k(y)} g\| \le e^{\alpha\delta(n-1)} \lambda^{n-1}.\end{aligned} \tag{5.33}$$

Provided that δ is sufficiently small, we have $\mu := e^{\alpha\delta}\lambda < 1$. By (5.31), (5.32), and (5.33), we obtain

$$\|d_x g^n (d_y g^n)^{-1}\| \le \|d_{g(x)} g^{n-1} (d_{g(y)} g^{n-1})^{-1}\| (1 + C\mu^{n-1}),$$

for some constant $C > 0$. By induction we conclude that

$$\begin{aligned}\|d_x g^n (d_y g^n)^{-1}\| &\le \|d_{g^{n-1}(x)} g (d_{g^{n-1}(y)} g)^{-1}\| \prod_{k=1}^{n-1} (1 + C\mu^k) \\ &\le D \prod_{k=1}^{\infty} (1 + C\mu^k) =: e^{\varepsilon},\end{aligned}$$

for some constants $D, \varepsilon > 0$. Therefore,

$$\|d_x g^n\| / \|d_y g^n\| \le \|d_x g^n (d_y g^n)^{-1}\| \le e^{\varepsilon},$$

and

$$\|(d_y g^n)^{-1}\| / \|(d_x g^n)^{-1}\| \le \|d_x g^n (d_y g^n)^{-1}\| \le e^{\varepsilon}.$$

This shows that both sequences $\underline{\Phi}^*$ and $\overline{\Phi}^*$ satisfy property (4.2), and that the inequalities in (5.30) hold. □

Under the hypotheses of Proposition 5.2.7, we have $\underline{s}^* = \underline{s}$ and $\overline{s}^* = \overline{s}$. Theorem 5.2.5 thus implies the following statement.

Corollary 5.2.8. *If g is a $C^{1+\alpha}$ expanding map with α-bunched derivative, for some $\alpha \in (0,1]$, and J is a repeller of g such that $g|J$ is topologically mixing, then*

$$\underline{s}^* \le \dim_H J \le \underline{\dim}_B J \le \overline{\dim}_B J \le \overline{s}^*. \tag{5.34}$$

In [59], Falconer considered C^2 expanding maps with 1-bunched derivative and obtained an upper estimate for $\overline{\dim}_B J$ that is in general sharper than that in Corollary 5.2.8 (see Theorem 5.3.2).

We note that there are nonconformal repellers for which the Hausdorff and box dimensions do not coincide. An example is described by Pollicott and Weiss in [158], modifying a construction of Przytcki and Urbański in [160] depending on delicate number-theoretical properties.

The following example illustrates that the numbers $\underline{s}$ and $\overline{s}$, and hence also the numbers $\underline{s}^*$ and $\overline{s}^*$, may provide sharper estimates for the dimension of the repeller than the numbers $\underline{t}$ and $\overline{t}$ in (5.22).

Example 5.2.9. There exists a repeller of a C^∞ expanding map with 1-bunched derivative, on a compact manifold, for which $\underline{t} < \underline{s} = \overline{s} < \overline{t}$.

Construction. We consider two tori $X_1 = X_2 = \mathbb{T}^2$ with the flat metric, and their disjoint union $X = X_1 \cup X_2$. We define a map $g\colon X \to X$ by $g(X_1) \subset X_2$, $g(X_2) \subset X_1$, and

$$g(x,y) = \begin{cases} (2x, 3y) & \text{if } (x,y) \in X_1, \\ (3x, 2y) & \text{if } (x,y) \in X_2. \end{cases}$$

Clearly, the map g is continuous and expanding. Since

$$\|d_p g^{2n}\| = 6^n \quad \text{and} \quad \prod_{k=0}^{2n-1} \|d_{g^k p} g\| = 3^{2n} = 9^n$$

for each $p \in X$, the map g is not conformal (see Definition 5.2.10). Now we consider the set $A \subset X = X_1 \cup X_2$ composed of the four rectangles (see Figure 5.1):

$$\left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{3}\right], \left[\frac{1}{2}, 1\right] \times \left[\frac{2}{3}, 1\right] \subset X_1$$

and

$$\left[0, \frac{1}{3}\right] \times \left[0, \frac{1}{2}\right], \left[\frac{2}{3}, 1\right] \times \left[\frac{1}{2}, 1\right] \subset X_2.$$

We also consider the repeller

$$J = \bigcap_{n=1}^{\infty} g^{-n} A$$

of the map g. By construction, for each $p \in X$ and $n \in \mathbb{N}$, we have

$$\left(\frac{2}{3}\right)^{1/2} \le 6^{n/2}\|d_p g^n\|^{-1} \le 1 \quad \text{and} \quad 3^n \prod_{k=0}^{n-1} \|d_{g^k p} g\|^{-1} = 1,$$

as well as

$$1 \le 6^{n/2}\|(d_p g^n)^{-1}\| \le \left(\frac{3}{2}\right)^{1/2} \quad \text{and} \quad 2^n \prod_{k=0}^{n-1} \|(d_{g^k p} g)^{-1}\| = 1.$$

The derivative of g is 1-bunched because

$$\|(d_p g)^{-1}\|^2 \|d_p g\| = 3/4 < 1$$

for every $p \in X$. Furthermore,

$$2\log 2/\log 3 = \underline{t} < \underline{s} = 4\log 2/\log 6 = \overline{s} < \overline{t} = 2.$$

This concludes the example. □

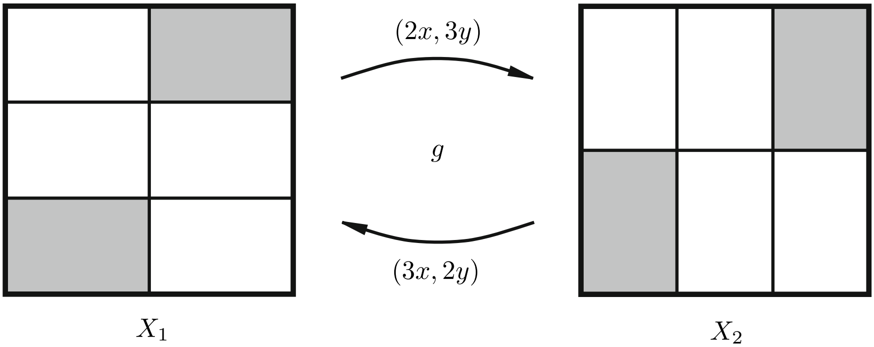

Figure 5.1: The set A is the union of the four gray rectangles

We note that for the particular construction in Example 5.2.9 it follows from Corollary 5.2.8 that the Hausdorff and box dimensions of the repeller coincide while this information cannot be obtained using only Theorem 5.2.4. This illustrates that one may obtain sharper dimension estimates or even exact values for the dimension using the nonadditive thermodynamic formalism instead of the classical thermodynamic formalism.

5.2.3 Conformal maps

We consider in this section the particular case of the repellers of a conformal map, that is, a map such that its differential is a multiple of an isometry at each point. In this case the dimension estimates in the former section become equalities.

The rigorous definition of conformal map is the following.

Definition 5.2.10. A map g is said to be *conformal* on a set J if $d_x g$ is a multiple of an isometry for every $x \in J$.

We first show that a conformal map is asymptotically conformal (in the sense of Definition 5.1.13).

Proposition 5.2.11. *Any C^1 conformal expanding map is asymptotically conformal.*

Proof. Let g be a C^1 conformal expanding map, and let J be a repeller of g. Let also $R_1, \ldots, R_\kappa$ be a Markov partition of J. For each $i = 1, \ldots, \kappa$, we take a sufficiently small open neighborhood $\hat{R}_i$ of R_i such that

$$\hat{R}_i \cap \hat{R}_j = \varnothing \quad \text{whenever} \quad R_i \cap R_j = \varnothing.$$

We also consider the sets $\hat{R}_{i_1 \cdots i_n}$ in (5.23).

Since J is compact, the map $x \mapsto \|d_x g\|$ is uniformly continuous. Hence, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\bigl|\log\|d_x g\| - \log\|d_y g\|\bigr| < \varepsilon \quad \text{whenever} \quad d(x,y) < \delta.$$

Moreover, we can always assume that $\operatorname{diam} \hat{R}_i < \delta$ for $i = 1, \ldots, \kappa$ (by eventually rechoosing the Markov partition).

Now we observe that

$$g^j(\hat{R}_{i_1 \cdots i_n}) \subset \hat{R}_{i_{j+1} \cdots i_n} \quad \text{for} \quad j = 0, \ldots, n-1.$$

Hence, $d(g^j(x), g^j(y)) < \delta$ for each $x, y \in \hat{R}_{i_1 \cdots i_n}$ and $j = 0, \ldots, n-1$. Since g is conformal, we obtain

$$\left|\log \frac{\|d_x g^n\|}{\|d_y g^n\|}\right| = \left|\sum_{j=0}^{n-1} \log \frac{\|d_{g^j(x)} g\|}{\|d_{g^j(y)} g\|}\right| \le n\varepsilon \tag{5.35}$$

for every $x, y \in \hat{R}_{i_1 \cdots i_n}$ and $n \in \mathbb{N}$. Moreover, we have $\|d_x g\|^{-1} = \|(d_x g^n)^{-1}\|$, again because g is conformal. By (5.35), we conclude that

$$e^{-n\varepsilon} \|d_x g^n\|^{-1} \le \underline{\lambda}_k(\omega, n) \le \overline{\lambda}_k(\omega, n) \le \|d_x g^n\|^{-1} e^{n\varepsilon}$$

for every $x = \chi(\omega) \in J$ and $n \in \mathbb{N}$. Therefore,

$$e^{-2n\varepsilon} \le \overline{\lambda}_k(\omega, n) / \underline{\lambda}_k(\omega, n) \le e^{2n\varepsilon},$$

and since ε is arbitrary, the map g is asymptotically conformal. □

When g is of class $C^{1+\alpha}$ for some $\alpha > 0$, the conclusion of Proposition 5.2.11 follows from the bounded distortion property (see for example [167]).

The following example shows that the converse of Proposition 5.2.11 does not hold.

Example 5.2.12. There is a C^∞ expanding map $g\colon X \to X$ of a compact manifold such that:

1. g is not conformal;
2. g is asymptotically conformal, and there exists $C > 0$ such that $\overline{\lambda}_k(\omega, n) \le C\underline{\lambda}_k(\omega, n)$ for every ω and $n, k \in \mathbb{N}$.

Construction. Let $g\colon X \to X$ be the C^∞ map constructed in Example 5.2.9. We choose a Markov partition of X such that each of its elements is contained either in X_1 or in X_2. Because of this, to compute the ratio coefficients it is sufficient to consider points only in one of the sets X_1 and X_2. Now we observe that there exist constants $C_1, C_2 > 0$ such that

$$C_1 6^{n/2}\|p-q\| \le \|g^n(p) - g^n(q)\| \le C_2 6^{n/2}\|p-q\|$$

for every $p, q \in X_1$ or $p, q \in X_2$. This implies that

$$\overline{\lambda}_k(\omega, n) \le (C_2/C_1)\underline{\lambda}_k(\omega, n)$$

for every ω and $n, k \in \mathbb{N}$. In particular, the map g is asymptotically conformal. □

Now we define a function $\varphi\colon M \to \mathbb{R}$ by

$$\varphi(x) = -\log\|d_x g\|.$$

Proposition 5.2.11 and Theorem 5.1.14 readily imply the following.

Theorem 5.2.13. *Let J be a repeller of a C^1 conformal expanding map g which is topologically mixing on J. For each open set U such that $J \cap U \ne \varnothing$, we have*

$$\dim_H(J\cap U) = \underline{\dim}_B(J\cap U) = \overline{\dim}_B(J\cap U) = t,$$

where t is the unique root of the equation $P_J(t\varphi) = 0$.

Proof. It follows from the proof of Proposition 5.2.11 that given $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that

$$-n\varepsilon + \sum_{j=0}^{n-1}\varphi(g^j(x)) \le \underline{\varphi}_{k,n}(\omega) \le \overline{\varphi}_{k,n}(\omega) \le n\varepsilon + \sum_{j=0}^{n-1}\varphi(g^j(x))$$

for every $x = \chi(\omega) \in J$ and $n \in \mathbb{N}$. By the first property in Theorem 4.2.2, these inequalities imply that

$$-\varepsilon + P_J(s\varphi) \le \overline{P}_{\Sigma_A^+}(s\underline{\Phi}_k) \le \overline{P}_{\Sigma_A^+}(s\overline{\Phi}_k) \le \varepsilon + P_J(s\varphi),$$

and

$$-\varepsilon + P_J(s\varphi) \le P_{\Sigma_A^+}(s\underline{\Phi}_k) \le P_{\Sigma_A^+}(s\overline{\Phi}_k) \le \varepsilon + P_J(s\varphi),$$

for every $s \geq 0$. Since ε is arbitrary, we thus obtain

$$\overline{P}_{\Sigma_A^+}(s\underline{\Phi}_k) = P_{\Sigma_A^+}(s\overline{\Phi}_k) = P_J(s\varphi).$$

The desired statement follows now immediately from Proposition 5.2.11 and Theorem 5.1.14. □

Ruelle [167] showed that $\dim_H J = s$ when f is of class $C^{1+\alpha}$. His proof consists of showing that the s-dimensional Hausdorff measure in J is equivalent to the equilibrium measure of $s\varphi$ (with Radon–Nikodym derivative bounded and bounded away from zero). The particular case of quasi-circles was earlier considered by Bowen in [40], where he also introduced the equation that now bears his name. The coincidence between the Hausdorff and box dimensions was established by Falconer in [57]. The extension of these results to repellers of C^1 maps was obtained independently by Barreira [5] and Gatzouras and Peres [73], using different approaches. This extension was obtained earlier by Takens in [187] for a class of Cantor sets. More recently, Rugh [168] gave a proof of Bowen's formula that also extends to time-dependent conformal repellers. We refer to [190] for a survey of the dimension theory of holomorphic endomorphisms.

5.3 Further developments

We want to give a panorama of the existing results concerning dimension estimates for a repeller when more geometric information is available, with emphasis on nonconformal repellers and on the relation to the thermodynamic formalism. We refer to the surveys [11, 43] for a fairly complete view of the state-of-the-art in the area. We shall follow closely [11].

5.3.1 Sharp upper dimension estimates

Given a linear map $L\colon \mathbb{R}^m \to \mathbb{R}^m$, let

$$\sigma_1(L) \geq \cdots \geq \sigma_n(L) \geq 0$$

be the *singular values* of L, that is, the eigenvalues of $(L^*L)^{1/2}$, counted with their multiplicities, where L^* denotes the transpose of L. These numbers coincide with the semiaxes of the ellipsoid which is the image of the unit ball in $\mathbb{R}^m$ under the map L. For each $s \in [0, n]$, we set

$$\omega_s(L) = \sigma_1(L) \cdots \sigma_{\lfloor s \rfloor}(L)\sigma_{\lfloor s \rfloor+1}(L)^{s-\lfloor s \rfloor}, \tag{5.36}$$

where $\lfloor s \rfloor$ denotes the integer part of s.

In [49], Douady and Oesterlé obtained an upper bound for the Hausdorff dimension in terms of the singular values. Namely, given a C^1 map $f\colon M \to M$, we consider the function $\psi_s\colon M \to \mathbb{R}$ defined by

$$\psi_s(x) = \log \omega_s(d_x f).$$

Moreover, given a set $J \subset M$, let

$$\dim_L(f, J) = \inf\left\{s \in (0, \dim M] : \sup_{x\in J} \psi_s(x) < 0\right\}.$$

Theorem 5.3.1 ([49]). *If f is a C^1 map and J is a compact f-invariant set, then*

$$\dim_H J \le \dim_L(f, J). \tag{5.37}$$

Since J is f^m-invariant for every $m \in \mathbb{N}$, we also have

$$\dim_H J \le \inf_{m\in\mathbb{N}} \dim_L(f^m, J), \tag{5.38}$$

which sometimes may give a better estimate than that in (5.37). It was shown by Hunt in [94] that we can replace $\dim_H J$ by $\overline{\dim}_B J$ in (5.37) for maps in $\mathbb{R}^n$. We refer to [74] for a proof including the case of maps on manifolds. Leonov [120] obtained estimates for the Hausdorff dimension under weaker assumptions using Lyapunov-type functions. Namely, to show that $\dim_H J \le s$ it is sufficient to show that

$$\sup_{x\in J}\left(\frac{\psi(f(x))}{\psi(x)}\omega_s(d_x f)\right) < 1,$$

where $\psi\colon J \to \mathbb{R}^+$ is some continuous function. This approach can be interpreted as a change of metric on the manifold, as studied for example by Noack and Reitmann in [142]. Sometimes it may improve the estimate in Theorem 5.3.1. Indeed, while the Hausdorff dimension is invariant under smooth changes of the metric, the singular values may change, and thus the function corresponding to the new metric may be strictly smaller than φ_s. We refer to [32, 74, 121] for more details.

Now we consider the sequences of functions $\Phi_s = (\varphi_{s,n})_{n\in\mathbb{N}}$ defined by

$$\varphi_{s,n}(x) = \log \omega_s((d_x f^n)^{-1}),$$

with ω_s as in (5.36). Using these functions Falconer [59] computed the dimension of a class of repellers of nonconformal transformations (building on his work [56]). His main result can be reformulated as follows.

Theorem 5.3.2. *If J is a repeller of a C^2 map f which is topologically mixing on J, and*

$$\|(d_x f)^{-1}\|^2 \|d_x f\| < 1 \quad \text{for every} \quad x \in J, \tag{5.39}$$

then

$$\underline{\dim}_B J = \overline{\dim}_B J \le s, \tag{5.40}$$

where s is the unique root of the equation $P_J(\Phi_s) = 0$.

Clearly, $\omega_s(L) \le \sigma_1(L)^s$, and hence,

$$\begin{aligned}\varphi_{s,n}(x) &\le s\log\sigma_1((d_x f^n)^{-1})\\ &\le s\log\|(d_x f^n)^{-1}\| \le s\sum_{k=0}^{n-1}\overline{\varphi}(f^k(x)),\end{aligned}$$

with $\overline{\varphi}$ as in (5.21). This shows that $s \le \overline{s}^*$, with $\overline{s}^*$ as in (5.34), and thus in general (5.40) may be a sharper estimate for the box dimension than that given by the upper bound in (5.34).

Under an additional geometric assumption, satisfied for example when J contains a nondifferentiable arc, the number s in Theorem 5.3.2 is equal to $\dim_H J$ (see [59]). Related results were obtained by Zhang in [202], and in the case of volume expanding maps by Gelfert in [75]. In particular Zhang proved the following.

Theorem 5.3.3. *If J is a repeller of a C^1 map f which is topologically mixing on J, then $\dim_H J \le s$, where s is the unique root of the equation $P_J(\Phi_s) = 0$.*

The formulation of this result in terms of the nonadditive topological pressure is due to Ban, Cao and Hu [4]. While Theorem 5.3.3 only gives an upper bound for the Hausdorff dimension, and not for the box dimension as in (5.40), on the other hand it does not require condition (5.39).

In another direction, Hu [92] computed the box dimension of a class of repellers of nonconformal transformations leaving invariant a strong unstable foliation. His formula for the box dimension is also expressed in terms of the topological pressure. Related results were obtained earlier by Bedford in [27] (see also [28]), for a class of self-similar sets that are graphs of continuous functions.

5.3.2 Self-affine repellers and nonlinear extensions

In another direction, Falconer [55, 58] studied a class of limit sets obtained from the composition of affine transformations that are not necessarily conformal. Consider affine transformations $f_i\colon \mathbb{R}^n \to \mathbb{R}^n$, for $i = 1, \dots, \kappa$, given by

$$f_i(x) = A_i x + b_i \tag{5.41}$$

for some linear contraction A_i and some vector $b_i \in \mathbb{R}^n$. We recall that A_i is said to be a *contraction* if there exists $c_i \in (0,1)$ such that $\|A_i\| < 1$. Then the following result of Hutchinson applies.

Theorem 5.3.4 ([97]). *Given C^1 maps $f_i\colon \mathbb{R}^n \to \mathbb{R}^n$, for $i = 1, \dots, \kappa$, let us assume that there exists $\lambda \in (0,1)$ such that*

$$\|f_i(x) - f_i(y)\| \le \lambda\|x - y\|$$

for every $x, y \in \mathbb{R}^n$ and $i = 1, \dots, \kappa$. Then there is a unique nonempty compact set $J \subset \mathbb{R}^n$ such that

$$J = \bigcup_{i=1}^{\kappa} f_i(J). \tag{5.42}$$

Moreover, for each nonempty compact set $R \subset \mathbb{R}^n$ such that $f_i(R) \subset R$ for $i = 1, \ldots, \kappa$, we have

$$J = \bigcap_{k=1}^{\infty} \bigcup_{i_1 \cdots i_k} (f_{i_1} \circ \cdots \circ f_{i_k})(R). \tag{5.43}$$

The set J in (5.43) is usually called the *limit set* of the family of maps $f_1, \ldots, f_\kappa$. We also consider the so-called *open set condition*: there is a nonempty open set U such that $f_i(U) \subset U$ for $i = 1, \ldots, \kappa$, and

$$f_i(U) \cap f_j(U) = \varnothing \quad \text{whenever} \quad i \neq j.$$

When U is a neighborhood of the set J, and the maps $f_1, \ldots, f_\kappa$ are invertible, one can define a map $g\colon \bigcup_{i=1}^{\kappa} f_i(U) \to U$ by $f|f_i(U) = f_i^{-1}$. Then g is a smooth expanding map, and the limit set J is a repeller of g.

Now we return to the class of maps in (5.41). In this case, the limit set J in Theorem 5.3.4 is called a *self-affine set* or a *self-affine repeller* (in case it is a repeller). In the particular case when each map A_i is a multiple of a rotation, the set J is also called a *self-similar set*. If, in addition, the open set condition holds, then the smooth map g constructed above is conformal. Let

$$\begin{aligned} s &= \inf\left\{ d \in [0,n] : \sum_{k=1}^{\infty} \sum_{i_1 \cdots i_k} \omega_d(A_{i_1} \circ \cdots \circ A_{i_k}) < \infty \right\} \\ &= \sup\left\{ d \in [0,n] : \sum_{k=1}^{\infty} \sum_{i_1 \cdots i_k} \omega_d(A_{i_1} \circ \cdots \circ A_{i_k}) = \infty \right\}, \end{aligned}$$

with ω_d as in (5.36). The number s is also uniquely determined by the condition

$$\lim_{k \to \infty} \frac{1}{k} \log \sum_{i_1 \cdots i_k} \omega_s(A_{i_1} \circ \cdots \circ A_{i_k}) = 0$$

(this identity is a particular case of Bowen's equation). We emphasize that s does not depend on the vectors $b_1, \ldots, b_\kappa$.

Theorem 5.3.5. *We have $\overline{\dim}_B J \le s$. In addition, if $\|A_i\| < 1/2$ for $i = 1, \ldots, \kappa$, then*

$$\dim_H J = \underline{\dim}_B J = \overline{\dim}_B J = s \tag{5.44}$$

for Lebesgue almost every $(b_1, \ldots, b_\kappa) \in (\mathbb{R}^n)^\kappa$.

The statement in Theorem 5.3.5 is due to Falconer [55] when $\|A_i\| < 1/3$ for $i = 1, \ldots, \kappa$, and to Solomyak [186] in the general case.

The class of self-affine repellers can also be used to illustrate that to determine or even to estimate the dimension of the limit set J, sometimes it is not sufficient to know the geometric shape of the sets $R_{i_1 \cdots i_n}$, in strong contrast to what happens in Theorem 5.2.13. For example, the dimension can be affected by certain number-theoretical properties. Namely, take $\kappa = 2$ and assume that the sets

$$R_{i_1 \cdots i_n} = (f_{i_1} \circ \cdots \circ f_{i_n})([0,1] \times [0,1])$$

are rectangles with sides of length a^n and b^n, obtained from the composition of the maps

$$f_1(x,y) = (ax, by) \quad \text{and} \quad f_2(x,y) = (ax - a + 1, by - b + 1),$$

for some fixed constants $a \in (0,1)$ and $b \in (0,1/2)$ with $b \leq a$. In particular, the projection of each set $R_{i_1 \cdots i_n}$ on the horizontal axis is an interval with right endpoint given by

$$a^n + \sum_{k=0}^{n-1} j_k a^k, \tag{5.45}$$

where

$$j_k = \begin{cases} 0 & \text{if } i_k = 1, \\ 1 - a & \text{if } i_k = 2. \end{cases}$$

Now we assume that $a = (\sqrt{5} - 1)/2$. In this case we have $a^2 + a = 1$, and thus, for each $n > 2$ there is more than one sequence $(i_1 \cdots i_n)$ with the same value in (5.45). This causes a larger concentration of the sets $R_{i_1 \cdots i_n}$ in certain regions of the limit set J. Therefore, to compute its Hausdorff dimension, when we take an open cover of J it may happen that it is possible to replace some elements of the cover by a single element. This may cause the set J to have a Hausdorff dimension strictly smaller than its box dimension. An example was described by Pollicott and Weiss in [158], modifying a construction of Przytcki and Urbański in [160] that depends on delicate number-theoretical properties. The same phenomenon was observed by Neunhäuserer in [141]. We note that the constant $a = (\sqrt{5} - 1)/2$ is only an example among many other possible values that lead to a similar phenomenon.

Pollicott and Weiss [158] considered the case when the sets $f_1([0,1] \times [0,1])$ and $f_2([0,1] \times [0,1])$ are disjoint, and they established the following result.

Theorem 5.3.6. *If $a \in (0,1/2)$, then*

$$\dim_H J = \underline{\dim}_B J = \overline{\dim}_B J = -\frac{\log 2}{\log a},$$

and if $a \in [1/2,1)$, then

$$\underline{\dim}_B J = \overline{\dim}_B J = 1 - \frac{\log(2a)}{\log b}.$$

They also showed that when $a \in [1/2,1)$ is a Garsia–Erdös number, then

$$\dim_H J = \underline{\dim}_B J = \overline{\dim}_B J = 1 - \frac{\log(2a)}{\log b}.$$

We recall that $a \in [0,1]$ is called a *Garsia–Erdös number* or simply a *GE number* if there exists $C > 0$ such that

$$\operatorname{card}\left\{(i_1,\ldots,i_n) \in \{0,1\}^n : \sum_{j=0}^{n-1} i_{j+1}a^j \in [x, x+a^n)\right\} \le C(2a)^n$$

for every $x \ge 0$. We note that:

1. there are no GE numbers in the interval $(0,1/2)$;
2. Lebesgue almost every $a \in (0,1/2)$ is a GE number;
3. not all numbers in $(1/2,1)$ are GE numbers: indeed, any reciprocal of a Pisot–Vijayarghavan number (that is, a root of an algebraic equation whose all conjugates have moduli less than 1) is not a GE number.

McMullen [135] and Gatzouras and Lalley [72] also obtained different Hausdorff and box dimensions for particular classes of self-affine sets (see in particular Theorem 5.3.11). Moreover, the s-dimensional Hausdorff measure with $s = \dim_H J$ need not be positive neither finite [72, 150]. Finally, for a smoothly parameterized family of self-affine sets the Hausdorff dimension may not vary continuously with the parameter [55, 72, 160].

On the other hand, Theorem 5.3.2 gives no indication of which sets J actually have dimension equal to s. In [93], Hueter and Lalley gave sufficient conditions for a given set $J \subset \mathbb{R}^2$ to satisfy (5.44). Let $f_1,\ldots,f_\kappa\colon \mathbb{R}^2 \to \mathbb{R}^2$ be again the affine transformations in Theorem 5.3.2, which we continue to write in the form $f_i(x) = A_i x + b_i$.

Theorem 5.3.7 ([93]). *Assume that:*

1. $\|A_i\| < 1$ *and* $\sigma_1(A_i)^2 < \sigma_2(A_i)$ *for* $i = 1,\ldots,\kappa$*;*
2. *the sets* $A_1^{-1}Q,\ldots,A_\kappa^{-1}Q$ *are disjoint, where* Q *is the closed second quadrant;*
3. *the sets* $A_1 J,\ldots,A_\kappa J$ *are disjoint.*

Then (5.44) *holds. Moreover, the* s*-dimensional Hausdorff measure of* J *is finite, and there is a unique ergodic invariant measure* μ *with* $\dim_H \mu = \dim_H J$*.*

We note that all the hypotheses in the theorem persist under sufficiently small perturbations of the entries of the matrices $A_1,\ldots,A_\kappa$.

A nonlinear generalization of the work of Hueter and Lalley was obtained by Luzia in [123]. Let $f_1,\ldots,f_\kappa\colon \mathbb{R}^2 \to \mathbb{R}^2$ be C^2 diffeomorphisms such that:

1. $\sup_{x\in\mathbb{R}^2} \|d_x f_i\| < 1$ for $i = 1,\ldots,\kappa$;
2. there is a convex bounded open set U such that $f_1(\overline{U}),\ldots,f_\kappa(\overline{U})$ are pairwise disjoint subsets of U;
3. $d_x f_i P \subset \operatorname{int} P$ for every $x \in U$, where P is the union of the closed first and third quadrants;

4. $\|d_x f_i v\|^3/|\det d_x f_i| < 1$ for every $x \in U$ and $v \in P$ with $\|v\| = 1$.

Then there is a unique nonempty compact set satisfying (5.42). We also consider the function $\varphi\colon \Sigma_\kappa^+ \to \mathbb{R}$ defined by

$$\varphi(i_1 i_2 \cdots) = \log \|d_{\pi(i_2 i_3 \cdots)} f_{i_1} | V\|,$$

where

$$\pi(i_2 i_3 \cdots) = \lim_{n\to\infty} (f_{i_2} \circ \cdots \circ f_{i_n})(\overline{U}),$$

and

$$V = \lim_{n\to\infty} d_{(f_{i_1} \circ \cdots \circ f_{i_n})^{-1} \pi(i_1 i_2 \cdots)} (f_{i_1} \circ \cdots \circ f_{i_n}) P.$$

Then the following statement holds.

Theorem 5.3.8 ([123]). *We have*

$$\dim_H J = \underline{\dim}_B J = \overline{\dim}_B J = s,$$

where s is the unique root of the equation $P_J(s\varphi) = 0$.

5.3.3 Measures of full dimension

We consider in this section the related problem of the existence of measures of full dimension in repellers. We first recall the definition.

Definition 5.3.9. An invariant measure μ in a repeller J is called a *measure of full dimension* if $\dim_H \mu = \dim_H J$.

It was shown by Ruelle in [167] that for a $C^{1+\alpha}$ map f which is conformal and topologically mixing on J, if μ is the unique equilibrium measure of the function $-\dim_H J \log\|d_x f\|$, then μ is a measure of full dimension. This follows from the equivalence between μ and the s-dimensional Hausdorff measure in J (see [7] for details). The existence of an ergodic measure of full dimension in any repeller of a C^1 map was established by Gatzouras and Peres in [73].

The situation is much more complicated in the case of nonconformal transformations, and there exist only some partial results. In particular, Gatzouras and Peres [73] considered maps of the form

$$f(x_1, x_2) = (f_1(x_1), f_2(x_2)),$$

where f_1 and f_2 are C^1 maps with repellers respectively J_1 and J_2 such that f_i is conformal on J_i for $i = 1, 2$. Then $f|(J_1 \times J_2)$ is a factor of a topological Markov chain, and we denote the factor map by π.

Theorem 5.3.10 ([73]). *If*

$$\min_{x_1 \in J_1} \|d_{x_1} f_1\| \geq \max_{x_2 \in J_2} \|d_{x_2} f_2\|,$$

then for any compact f-invariant set $J \subset J_1 \times J_2$ such that $\pi^{-1}J$ has the specification property, we have

$$\dim_H J = \sup_\mu \dim_H \mu,$$

with the supremum taken over all ergodic f-invariant probability measures in J.

For piecewise linear maps f_i, Gatzouras and Lalley [72] showed earlier that certain invariant sets, corresponding to full shifts in the symbolic dynamics, carry an ergodic measure of full dimension (they also obtained implicit formulas for the Hausdorff and box dimensions of the invariant sets, as well as conditions for their coincidence). Kenyon and Peres [110] obtained the same result for linear maps f_i and arbitrary compact invariant sets (again with formulas for the Hausdorff and box dimensions of the invariant sets, as well as conditions for their coincidence). The proof of Theorem 5.3.10 uses the result in [72] and the idea from [110] of approximating arbitrary invariant sets by self-affine sets (corresponding to full shifts in the symbolic dynamics). Bedford and Urbański considered a particular class of self-affine sets in [28] and obtained conditions for the existence of a measure of full dimension.

Earlier related ideas appeared in work of Bedford [26] and McMullen [135]. We briefly describe some of their results. Given integers $m \le n$ and a set

$$D = \{d_j : j = 1, \dots, \kappa\} \subset \{0, \dots, n-1\} \times \{0, \dots, m-1\},$$

we define

$$J = \left\{ \sum_{j=1}^\infty \begin{pmatrix} n^{-j} & 0 \\ 0 & m^{-j} \end{pmatrix} d : d \in D \text{ for each } k \in \mathbb{N} \right\}. \tag{5.46}$$

The set J is called a *general Sierpiński carpet*. We note that viewed as a subset of the 2-torus $\mathbb{T}^2$ the set J is invariant under the map $\left(\begin{smallmatrix} n & 0 \\ 0 & m \end{smallmatrix}\right)$. Moreover, J is the self-affine set obtained from the affine transformations

$$f_i(x, y) = \left(\frac{x + k_i}{n}, \frac{y + l_i}{m} \right), \quad i = 1, \dots, \kappa,$$

where $(k_i, l_i) = d_i$ for $i = 1, \dots, \kappa$. A coding map $\chi \colon \{1, \dots, \kappa\}^{\mathbb{N}} \to J$ is given by

$$\chi(i_1 i_2 \cdots) = \left(\sum_{j=1}^\infty k_{i_j} n^{-j}, \sum_{j=1}^\infty l_{i_j} m^{-j} \right).$$

Theorem 5.3.11 ([135]). *We have*

$$\dim_H J = \log_m \sum_{l=0}^{m-1} t_l^{\log_n m},$$

where t_l is the number of integers k such that $(k,l) \in D$, and

$$\underline{\dim}_B J = \overline{\dim}_B J = \log_m s + \log_n \frac{\kappa}{s},$$

where s is the number of integers l such that $(k,l) \in D$ for some k.

One can show that the Hausdorff and box dimensions of J coincide only when $n = m$ (in which case J is a repeller of a conformal map), or when t_l takes only one value besides zero.

Bedford and McMullen obtained the following result independently for the set J in (5.46).

Theorem 5.3.12. *There exists an ergodic measure of full dimension in J.*

The following higher-dimensional generalization was obtained by Kenyon and Peres [110].

Theorem 5.3.13. *If $J \subset \mathbb{T}^m$ is invariant under a toral endomorphism whose eigenvalues are roots of integers, then there exists an ergodic measure of full dimension in J.*

More recently, Yayama [198] considered general Sierpiński carpets modeled by arbitrary topological Markov chains, and gave conditions for the existence of a unique measure of full dimension. In another direction, Luzia [124] considered expanding maps of the 2-torus of the form

$$f(x,y) = (a(x,y), b(y))$$

that are C^2-perturbations of linear maps. He showed that if f is sufficiently C^2-close to a general Sierpiński carpet, then

$$\dim_H J = \sup_\mu \dim_H \mu, \tag{5.47}$$

where the supremum is taken over all ergodic f-invariant probability measures in J. He also showed in [125] that the supremum in (5.47) is attained.

5.3.4 Nonuniformly expanding repellers

There are only a few related results in the literature for transformations that are not uniformly expanding.

We first formulate a result of Gelfert in [75]. For each $s \in [0, \dim M]$, let

$$\varphi_s(x) = \log \omega_s((d_x f)^{-1}),$$

with ω_s as in (5.36).

Theorem 5.3.14. *If J is a compact invariant set of an expansive C^1 local diffeomorphism f, then*

$$\overline{\dim}_B J \le \inf \left\{ s \in (0, \dim M] \colon \sup_{x \in J} \varphi_s(x) < 1 \text{ and } P_J(\varphi_s) < 0 \right\}.$$

We note that the proof of the corresponding result in [75] contains a mistake, although the statement is true under the additional hypothesis of expansivity.

In a different direction, Horita and Viana [90] and Dysman [50] studied abstract models, called maps with holes, which include examples of nonuniform repellers. Namely, let $f \colon M \to M$ be a map such that there exist domains (that is, compact path-connected sets) $R_1, \ldots, R_\kappa \subset M$ with pairwise disjoint interiors, such that $\overline{\dim}_B \partial R_i < \dim M$ for $i = 1, \ldots, p$. Moreover, we assume that each restriction $f|R_i$ is a $C^{1+\alpha}$ diffeomorphism onto some domain W_i containing $R_1 \cup \cdots \cup R_\kappa$, that the holes

$$H_i = W_i \setminus (R_1 \cup \cdots \cup R_\kappa)$$

have nonempty interior, and that the inner diameter (the supremum of the inner distances between any two points in the same connected component) of each set R_i is finite. The *repeller* of a map f with holes is the set of points whose forward orbit never falls into the holes, that is,

$$J = \left\{ x \in M \colon f^k(x) \in R_1 \cup \cdots \cup R_\kappa \text{ for } k \in \mathbb{N} \right\}.$$

Given $n \in \mathbb{N}$ and $i_1, \ldots, i_n \in \{1, \ldots, \kappa\}$, we set

$$C_{i_1 \cdots i_n} = \bigcap_{k=0}^{n-1} f^{-k} R_{i_{k+1}},$$

and

$$\varphi_n(i_1 \cdots i_n) = \frac{1}{n} \sum_{k=1}^{n} \inf_{x \in C_{i_1 \cdots i_k}} \log \| d_{f^k(x)} f^{-1} \|^{-1}.$$

Now we assume that there exist constants $\beta, \gamma > 0$ such that

$$\sum_{i_1 \cdots i_n} m(C_{i_1 \cdots i_n}) \le e^{-\beta n}$$

for any sufficiently large n, where the sum is taken over all sequences $(i_1 \cdots i_n)$ such that $\varphi_n(i_1 \cdots i_n) \le \gamma$, and where m denotes the Lebesgue measure. That is, although the expansion in J need not be uniform, the measure of the set of points remaining in a small neighborhood of J decreases exponentially fast in time. Under these assumptions, Dysman [50] showed that $\overline{\dim}_B J < \dim M$. This inequality generalizes a corresponding result for the Hausdorff dimension obtained by Horita and Viana in [90] under the same assumptions.

In [91], Horita and Viana studied nonuniformly expanding repellers emerging from a perturbation of an Anosov diffeomorphism f of the 3-torus through a Hopf bifurcation. A saddle point becomes an attractor, and the complement of its basin of attraction can be considered a repeller for the new diffeomorphism. We note that since the repeller contains an invariant circle, due to the bifurcation, it is not uniformly expanding. Building on the above result of Dysman it is shown in [91] that the box dimension of the repeller is strictly less than 3 for all diffeomorphisms sufficiently C^5-close to f but different from f.

5.3.5 Upper bounds involving Lyapunov exponents

Now we describe briefly some upper dimension estimates involving Lyapunov exponents. Related ideas go back to Kaplan and Yorke in [107].

The *local Lyapunov exponents* $\nu_1(x) \geq \cdots \geq \nu_n(x)$ of a smooth transformation $f\colon M \to M$ of an n-dimensional manifold are defined recursively by

$$\nu_1(x) + \cdots + \nu_j(x) = \limsup_{m\to\infty} \frac{1}{m} \log \omega_j(d_x f^m), \qquad j = 1, \ldots, n.$$

Now let μ be an ergodic f-invariant probability measure in M. Then the local Lyapunov exponents are constant μ-almost everywhere, and we denote them respectively by $\nu_1(\mu) \geq \cdots \geq \nu_n(\mu)$. The *Lyapunov dimension* with respect to μ is defined by

$$\dim_L(f, \mu) = k + \frac{\nu_1(\mu) + \cdots + \nu_k(\mu)}{|\nu_{k+1}(\mu)|},$$

where $k < n$ is the smallest integer such that $\nu_1(\mu) + \cdots + \nu_{k+1}(\mu) < 0$. Ledrappier established the following statement in [118].

Theorem 5.3.15. *If J is a compact f-invariant set, then*

$$\sup_\mu \dim_L(f, \mu) = \inf_{m\in\mathbb{N}} \dim_L(f^m, J),$$

with the supremum taken over all ergodic f-invariant probability measures in J.

It thus follows from (5.38) that

$$\dim_H J \leq \sup_\mu \dim_L(f, \mu). \tag{5.48}$$

There are examples where the supremum in (5.48) coincides with the Hausdorff dimension. Namely, let us consider a C^2 diffeomorphism f of a compact manifold and an ergodic f-invariant probability measure μ in M. It follows from work of Ledrappier and Young in [119] that if μ is an SRB measure, then

$$\dim_H J = \sup_\mu \dim_L(f, \mu). \tag{5.49}$$

However, in general (5.49) does not hold (we refer to [11] for an example).

Now we consider the so-called *uniform Lyapunov exponents* $\nu_1^u \geq \cdots \geq \nu_n^u$, defined recursively by

$$\nu_1^u + \cdots + \nu_j^u = \lim_{m\to\infty} \frac{1}{m} \log \max_{x\in J} \omega_j(d_x f^m), \qquad j = 1, \ldots, n.$$

We also define

$$d^u(f, J) = k + \frac{\nu_1^u + \cdots + \nu_k^u}{|\nu_{k+1}^u|},$$

where $k < n$ is the smallest integer such that $\nu_1^u + \cdots + \nu_{k+1}^u < 0$. We note that

$$d^u(f, J) \leq \inf_{m\in\mathbb{N}} \dim_L(f^m, J).$$

Temam [189] obtained the following result.

Theorem 5.3.16. *If f is a C^1 map, and J is a compact f-invariant set, then*

$$\overline{\dim}_B \Lambda \leq d^u(f, J).$$

We also refer to [189] for generalizations to the infinite-dimensional setting, and for applications to attractors of many physical systems. See [51, 54, 52, 53] for further developments. Related problems were considered by Blinchevskaya and Ilyashenko [31] and Chapyzhov and Ilyin [44].

Chapter 6

Dimension Estimates for Hyperbolic Sets

We consider in this chapter the dimension of hyperbolic sets, which are invariant sets of a hyperbolic invertible dynamics. The main aim is to develop as much as possible a corresponding theory to that in Chapter 5 in the case of repellers. In particular, after describing how Markov partitions can be used to model hyperbolic sets, we present several applications of the nonadditive thermodynamic formalism to the study of their dimension. In particular, we obtain lower and upper dimension estimates for a large class of hyperbolic sets, also of maps that need not be differentiable. In addition, besides deriving as a simple consequence corresponding results in the case of conformal dynamics, we survey the existing results for nonconformal dynamics and nonuniformly hyperbolic dynamics, in particular in what concerns obtaining sharper dimension estimates when further geometric information is available. In comparison to the case of repellers, the study of the dimension of hyperbolic sets presents an additional difficulty. Namely, a priori the dimensions in the stable and unstable directions may not add to give the dimension of the whole set, also depending on the regularity properties of the holonomies. For completeness, we include a discussion of the most recent results in the area.

6.1 Hyperbolic sets for homeomorphisms

We first consider the general case of hyperbolic sets for continuous maps. The main aim is to describe several applications of the nonadditive thermodynamic formalism to the study of the dimension of hyperbolic sets in this general setting.

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6.1.1 Basic notions

We first introduce some basic concepts, starting with the notions of stable and unstable sets and of Axiom $A^\sharp$ homeomorphism.

Let $h\colon X \to X$ be a homeomorphism of the compact metric space (X, d). Given $x \in X$ and $\varepsilon > 0$, we define the *local stable* and *unstable sets* (of size ε) at x respectively by

$$V^s(x) = \{y \in X : d(h^n(x), hf^n(y)) < \varepsilon \text{ for all } n \geq 0\},$$

and

$$V^u(x) = \{y \in X : d(h^n(x), h^n(y)) < \varepsilon \text{ for all } n \leq 0\}.$$

Definition 6.1.1. We say that h is an *Axiom $A^\sharp$ homeomorphism* if there exist constants $0 < \lambda < 1$ and $\varepsilon > 0$ such that:

1. for each $x \in X$, we have

$$d(h^n(y), h^n(z)) \leq \lambda^n d(y, z) \quad \text{for all} \quad y, z \in V^s(x), n \geq 0$$

and

$$d(h^n(y), h^n(z)) \leq \lambda^n d(y, z) \quad \text{for all} \quad y, z \in V^u(x), n \leq 0;$$

2. there exists $\delta > 0$ such that for each $x, y \in X$ with $d(x, y) < \delta$, the set $V^s(x) \cap V^u(y)$ consists of a single point, which we denote by $[x, y]$;
3. the map

$$[\cdot, \cdot]\colon \{(x, y) \in X \times X : d(x, y) < \delta\} \to X$$

is continuous.

This notion was introduced by Alekseev and Yakobson in [2], although it is formally contained in former work of Bowen. One can verify that h is an Axiom $A^\sharp$ homeomorphism if and only if X is a Smale space with respect to h and $[\cdot, \cdot]$ (in the sense of [166]).

Now we recall the notion of nonwandering point.

Definition 6.1.2. A point $x \in X$ is called a *nonwandering point* of h if for each open neighborhood U of x, there exists $n \in \mathbb{N}$ such that $h^n(U) \cap U \neq \varnothing$.

We denote by $\Omega(h)$ the set of nonwandering points of h. This is a closed h-invariant set. For an Axiom $A^\sharp$ homeomorphism, the set $\Omega(h)$ coincides with the closure of the set of periodic points of h. This follows from Bowen's proof of the shadowing lemma for Axiom A diffeomorphisms (see [39]).

Moreover, one can show that if h is an Axiom $A^\sharp$ homeomorphism, then there is a decomposition of $\Omega(h)$ into a finite number of disjoint closed h-invariant sets

$$\Omega(h) = \Lambda_1 \cup \cdots \cup \Lambda_m,$$

such that $h|\Lambda_i$ is topologically transitive for $i = 1, \dots, m$. This follows from Smale's proof of the so-called spectral decomposition of the nonwandering set of an Axiom A diffeomorphism. Each set Λ_i is called a *basic set* for h, and it is the union of a finite number n_i of disjoint compact sets $\Lambda_{i1}, \dots, \Lambda_{in_i}$ that are cyclically permuted by h, and such that $h^{n_i}|\Lambda_{ij}$ is topologically mixing for $j = 1, \dots, n_i$. Hence, in a similar manner to that for repellers in Section 5.1, without loss of generality we may assume that h is topologically mixing.

A nonempty closed set $R \subset X$ is called a *rectangle* if $\operatorname{diam} R < \delta$ (with δ as in Definition 6.1.1), $R = \overline{\operatorname{int} R}$, and $[x, y] \in R$ whenever $x, y \in R$. For each $x \in R$, we write

$$V^s(x, R) = V^s(x) \cap R \quad \text{and} \quad V^u(x, R) = V^u(x) \cap R.$$

Now let $\Lambda \subset X$ be a closed h-invariant set.

Definition 6.1.3. A finite cover of Λ by rectangles $R_1, \dots, R_\kappa$ is called a *Markov partition* of Λ (with respect to h) if:

1. $R_i \cap R_j \subset \partial R_i \cap \partial R_j$ whenever $i \neq j$;
2. for each $x \in \operatorname{int} R_i \cap h^{-1}(\operatorname{int} R_j)$, we have

$$h(V^u(x, R_i)) \supset V^u(h(x), R_j) \quad \text{and} \quad h(V^s(x, R_i)) \subset V^s(h(x), R_j).$$

Basic sets of Axiom $A^\sharp$ homeomorphisms have Markov partitions of arbitrarily small diameter (see [2, 166]), and these give rise to symbolic models for each basic set. Namely, let h be an Axiom $A^\sharp$ homeomorphism, and let Λ be a basic set for h. We consider a Markov partition $R_1, \dots, R_\kappa$ of Λ with diameter at most δ (with δ as in Definition 6.1.1), and we define a $\kappa \times \kappa$ matrix $A = (a_{ij})$ by

$$a_{ij} = \begin{cases} 1 & \text{if } \operatorname{int} R_i \cap h^{-1}(\operatorname{int} R_j) \neq \varnothing, \\ 0 & \text{if } \operatorname{int} R_i \cap h^{-1}(\operatorname{int} R_j) = \varnothing. \end{cases}$$

We also consider the shift map $\sigma\colon \Sigma_\kappa \to \Sigma_\kappa$ (see Section 3.2), and the (two-sided) topological Markov chain $\sigma|\Sigma_A$ with transition matrix A (see Definition 3.4.7). For each $\omega = (\cdots i_{-1} i_0 i_1 \cdots) \in \Sigma_A$, let

$$\chi(\omega) = \bigcap_{n=-\infty}^{\infty} h^{-n} R_{i_n}.$$

One can show that the set $\chi(\omega)$ consists of a single point in Λ, and thus, we obtain a *coding map* $\chi\colon \Sigma_A \to \Lambda$ for the basic set. Moreover, the following properties hold:

1. χ is onto and is Hölder continuous with respect to the distance d_β in (3.7);
2. $\chi \circ \sigma = h \circ \chi$, that is, we have the commutative diagram

$$\begin{array}{ccc} \Sigma_A & \xrightarrow{\sigma} & \Sigma_A \\ {\scriptstyle\chi}\big\downarrow & & \big\downarrow{\scriptstyle\chi} \\ \Lambda & \xrightarrow{h} & \Lambda \end{array}.$$

The map χ need not be invertible, although $\operatorname{card}\chi^{-1}x \le \kappa^2$ for every $x \in \Lambda$. For each $\omega = (\cdots i_{-1}i_0i_1\cdots) \in \Sigma_A$ and $n \in \mathbb{N}$, let

$$R^u_{i_0\cdots i_n} = \bigcap_{k=0}^{n} h^{-k}R_{i_k} \quad \text{and} \quad R^s_{i_{-n}\cdots i_0} = \bigcap_{k=-n}^{0} h^{-k}R_{i_k}.$$

Now we fix $x \in \Lambda \cap R_{k_0}$. For each $\omega = (\cdots i_{-1}i_0i_1\cdots) \in \Sigma_A$ with $i_0 = k_0$, and $n, k \in \mathbb{N}$, we define ratio coefficients in the unstable direction by

$$\underline{\lambda}^u_k(i_0i_1\cdots, n) = \min\inf\left\{\frac{d(y,z)}{d(h^n(y), h^n(z))} : y, z \in A^u \text{ and } y \ne z\right\}$$

and

$$\overline{\lambda}^u_k(i_0i_1\cdots, n) = \max\sup\left\{\frac{d(y,z)}{d(h^n(y), h^n(z))} : y, z \in A^u \text{ and } y \ne z\right\},$$

where

$$A^u = V^u(x, R^u_{j_0\cdots j_{n+k}}),$$

with the minimum and maximum taken over all Σ_A-admissible finite sequences $(j_0\cdots j_{n+k})$ such that $(j_0\cdots j_n) = (i_0\cdots i_n)$. Similarly, we define ratio coefficients in the stable direction by

$$\underline{\lambda}^s_k(i_0i_{-1}\cdots, n) = \min\inf\left\{\frac{d(y,z)}{d(h^{-n}(y), h^{-n}(z))} : y, z \in A^s \text{ and } y \ne z\right\}$$

and

$$\overline{\lambda}^s_k(i_0i_{-1}\cdots, n) = \max\sup\left\{\frac{d(y,z)}{d(h^{-n}(y), h^{-n}(z))} : y, z \in A^s \text{ and } y \ne z\right\},$$

where

$$A^s = V^s(x, R^s_{j_{-(n+k)}\cdots j_0}),$$

with the minimum and maximum taken over all Σ_A-admissible finite sequences $(j_{-(n+k)}\cdots j_0)$ such that $(j_{-n}\cdots j_0) = (i_{-n}\cdots i_0)$.

6.1.2 Dimension estimates

We obtain in this section dimension estimates for basic sets of Axiom $A^\sharp$ homeomorphisms. The main tool is again the nonadditive thermodynamic formalism developed in Chapter 4, together with the symbolic dynamics associated to the Markov partitions.

Let Λ be a basic set of an Axiom $A^\sharp$ homeomorphism $h\colon X \to X$. We always assume that the following property holds:

$$\text{the maps } h|V^u(x) \text{ and } h^{-1}|V^s(x) \text{ are expanding for every } x \in \Lambda. \tag{6.1}$$

More precisely, this means that the maps

$$h|V^u(x)\colon V^u(x) \to h(V^u(x)) \quad \text{and} \quad h^{-1}|V^s(x)\colon V^s(x) \to h^{-1}(V^s(x)) \tag{6.2}$$

satisfy the inclusions in (5.1), with h replaced by h^{-1} in the second case, for some constants

$$a_u(x) \ge b_u(x) > 1 \quad \text{and} \quad a_s(x) \ge b_s(x) > 1,$$

such that

$$\mu_u(x) = \limsup_{n\to\infty} \big(a_u(h^n(x)) \cdots a_u(x)\big)^{1/n} < \infty$$

and

$$\mu_s(x) = \limsup_{n\to\infty} \big(a_s(h^{-n}(x)) \cdots a_s(x)\big)^{1/n} < \infty$$

for every $x \in \Lambda$.

Given $x \in \Lambda \cap R_{k_0}$, we define

$$C^u(x) = \big\{(i_0 i_1 \cdots) \in \Sigma_A^+ : i_0 = k_0\big\} \tag{6.3}$$

and

$$C^s(x) = \big\{(i_0 i_{-1} \cdots) \in \Sigma_{A^*}^+ : i_0 = k_0\big\}, \tag{6.4}$$

where A^* denotes the transpose of A. We also define two sequences of functions in $C^u(x)$ by

$$\underline{\varphi}^u_{k,n}(\omega) = \log \underline{\lambda}^u_k(\omega, n) \quad \text{and} \quad \overline{\varphi}^u_{k,n}(\omega) = \log \overline{\lambda}^u_k(\omega, n),$$

as well as two sequences of functions in $C^s(x)$ by

$$\underline{\varphi}^s_{k,n}(\omega) = \log \underline{\lambda}^s_k(\omega, n) \quad \text{and} \quad \overline{\varphi}^s_{k,n}(\omega) = \log \overline{\lambda}^s_k(\omega, n).$$

By (6.1), a slight modification of the proof of Proposition 5.1.4 shows that for each $\varepsilon > 0$ we have

$$(\mu_u(x) + \varepsilon)^{-n} \le \underline{\lambda}^u_k(\omega, n) \le \overline{\lambda}^u_k(\omega, n) \le \lambda^n, \tag{6.5}$$

$$(\mu_s(x) + \varepsilon)^{-n} \le \underline{\lambda}^s_k(\omega, n) \le \overline{\lambda}^s_k(\omega, n) \le \lambda^n, \tag{6.6}$$

for every $n \in \mathbb{N}$ and all sufficiently large $k \in \mathbb{N}$.

By (6.5), it follows from the second property in Theorem 4.4.1 that there exist unique roots $\underline{r}^u_k(x)$ and $\overline{r}^u_k(x)$ respectively of the equations

$$\overline{P}_{C^u(x)}(r\underline{\Phi}^u_k) = 0 \quad \text{and} \quad P_{C^u(x)}(r\overline{\Phi}^u_k) = 0,$$

and similarly, by (6.6), there exist unique roots $\underline{r}^s_k(x)$ and $\overline{r}^s_k(x)$ respectively of the equations

$$\overline{P}_{C^s(x)}(r\underline{\Phi}^s_k) = 0 \quad \text{and} \quad P_{C^s(x)}(r\overline{\Phi}^s_k) = 0.$$

The following result gives dimension estimates for the intersection of the stable and unstable sets with a basic set, and can be obtained from a slight modification of the proof of Theorem 5.1.7.

Theorem 6.1.4 ([5]). *Let Λ be a basic set of a topologically mixing Axiom $A^\sharp$ homeomorphism. Then*

$$\sup_{k\in\mathbb{N}} \underline{r}_k^u(x) \le \dim_H V^u(x,\Lambda) \le \underline{\dim}_B V^u(x,\Lambda) \le \overline{\dim}_B V^u(x,\Lambda) \le \inf_{k\in\mathbb{N}} \overline{r}_k^u(x)$$

and

$$\sup_{k\in\mathbb{N}} \underline{r}_k^s(x) \le \dim_H V^s(x,\Lambda) \le \underline{\dim}_B V^s(x,\Lambda) \le \overline{\dim}_B V^s(x,\Lambda) \le \inf_{k\in\mathbb{N}} \overline{r}_k^s(x).$$

We emphasize that the dimensions in Theorem 6.1.4 may vary from point to point. However, property (6.1) implies that the maps in (6.2) are locally bi-Lipschitz (compare with Proposition 5.1.4). Hence, the Hausdorff and box dimensions of the stable and unstable sets remain the same along orbits, that is, we have

$$\dim_H V^u(h^n(x),\Lambda) = \dim_H V^u(x,\Lambda)$$

and

$$\dim_H V^s(h^n(x),\Lambda) = \dim_H V^s(x,\Lambda)$$

for every $n \in \mathbb{N}$, with similar identities for the lower and upper box dimensions.

For an Axiom $A^\sharp$ homeomorphism with a bi-Lipschitz map $(x,y) \mapsto [x,y]$, one can show that

$$\begin{aligned}\underline{r}_k^u(x) + \underline{r}_k^s(x) &\le \dim_H \big[V^u(x,\Lambda), V^s(x,\Lambda)\big]\\ &\le \underline{\dim}_B \big[V^u(x,\Lambda), V^s(x,\Lambda)\big]\\ &\le \overline{\dim}_B \big[V^u(x,\Lambda), V^s(x,\Lambda)\big] \le \overline{r}_k^u(x) + \overline{r}_k^s(x)\end{aligned}$$

for all sufficiently large $k \in \mathbb{N}$. This follows from a slight modification of the proof of Theorem 6.2.9.

Property (6.1) is automatically satisfied when h is of class C^1, but it may not hold for an arbitrary Axiom $A^\sharp$ homeomorphism. More generally, we have the following criterion of which the proof is straightforward.

Proposition 6.1.5. *If the maps h and h^{-1} are locally Lipschitz, then property* (6.1) *holds. In particular, $\mu_u(x), \mu_s(x) < \infty$ for every $x \in \Lambda$.*

In an analogous manner to that in Definition 5.1.13, we introduce the notion of asymptotic conformality for an Axiom $A^\sharp$ homoeomorphism.

Definition 6.1.6. The homeomorphism h is said to be *asymptotically conformal* on the set $V^u(x,\Lambda)$ if there exists $k \in \mathbb{N}$ such that

$$\frac{1}{n}\log\frac{\overline{\lambda}_k^u(\omega,n)}{\underline{\lambda}_k^u(\omega,n)} \to 0 \text{ uniformly on } \Sigma_A^+ \text{ when } n\to\infty.$$

The following result can be obtained from a slight modification of the proof of Theorem 5.1.14.

Theorem 6.1.7 ([5]). *Let Λ be a basic set of a topologically mixing Axiom $A^\sharp$ homeomorphism which is asymptotically conformal on $V^u(x,\Lambda)$. Then, for all sufficiently large $k \in \mathbb{N}$, we have*

$$\dim_H V^u(x,\Lambda) = \underline{\dim}_B V^u(x,\Lambda) = \overline{\dim}_B V^u(x,\Lambda) = \underline{r}_k^u(x) = \overline{r}_k^u(x) = r^u(x),$$

where $r^u(x)$ is the unique root of the equation

$$\lim_{n\to\infty} \frac{1}{n} \log \sum_{(i_0\cdots i_{n-1})\in S_n} \left(\operatorname{diam} V^u(x, R^u_{i_0\cdots i_{n-1}})\right)^r = 0,$$

and where S_n is the set of all Σ_A^+-admissible sequences of length n.

There is an analogous version of Theorem 6.1.7 for the stable sets $V^s(x,\Lambda)$.

6.2 Hyperbolic sets for diffeomorphisms

Now we consider hyperbolic sets for diffeomorphisms, and again we obtain corresponding dimension estimates using the nonadditive thermodynamic formalism. As in the case of repellers, some of the results are obtained as a consequence of the corresponding results for homeomorphisms. We emphasize that in general the diffeomorphisms that we consider are only of class C^1.

6.2.1 Basic notions

We first recall the notion of hyperbolic set for a diffeomorphism. We then describe several additional properties of the hyperbolic sets. Let $f\colon M \to M$ be a C^1 diffeomorphism, and let $\Lambda \subset M$ be a compact f-invariant set.

Definition 6.2.1. The set Λ is said to be a *hyperbolic set* for f if for every point $x \in \Lambda$ there exists a decomposition of the tangent space

$$T_xM = E^s(x) \oplus E^u(x)$$

such that

$$d_xfE^s(x) = E^s(f(x)) \quad \text{and} \quad d_xfE^u(x) = E^u(f(x)),$$

and there exist constants $\lambda \in (0,1)$ and $c > 0$ (independent of x) such that

$$\|d_xf^n|E^s(x)\| \le c\lambda^n \quad \text{and} \quad \|d_xf^{-n}|E^u(x)\| \le c\lambda^n$$

for every $x \in \Lambda$ and $n \in \mathbb{N}$.

One can show that the stable and unstable subspaces $E^s(x)$ and $E^u(x)$ vary continuously with $x \in \Lambda$.

The following are examples of hyperbolic sets.

Example 6.2.2 (Linear horseshoes in $\mathbb{R}^2$). Let f be a C^1 diffeomorphism contracting the unit square $Q = [0,1]^2 \subset \mathbb{R}^2$ horizontally by a factor $\lambda < 1/2$, expanding it vertically by a factor $1/\lambda$, and folding the resulting rectangle into a horseshoe. We assume that the set $f(Q) \cap Q$ is composed of two vertical strips V_1 and V_2, and that the set $Q \cap f^{-1}(Q)$ is composed of two horizontal strips H_1 and H_2. The compact f-invariant set $\Lambda \subset Q$ defined by

$$\Lambda = \bigcap_{n \in \mathbb{Z}} f^n Q$$

is a hyperbolic set for f with respect to the decomposition of the tangent space into horizontal and vertical lines $T_x\mathbb{R}^2 = E^s(x) \oplus E^u(x)$, where $E^s(x) = \mathbb{R}$ is the horizontal line and $E^u(x) = \mathbb{R}$ is the vertical line, for each $x \in \Lambda$. The set Λ is called a *Smale horseshoe*.

Example 6.2.3 (Solenoids). Let D be the closed unit disc in $\mathbb{R}^2$ centered at the origin. We define a map $f\colon S^1 \times D \to S^1 \times D$ by

$$f(\theta, x, y) = (2\theta \bmod 1, \lambda x + \varepsilon \cos(2\pi\theta), \mu y + \varepsilon \sin(2\pi\theta)),$$

for some constants $\lambda, \mu < \min\{\varepsilon, 1/2\}$. The compact f-invariant set

$$\Lambda = \bigcap_{n \in \mathbb{N}} f^n(S^1 \times D)$$

is a hyperbolic set for f, called a *solenoid*. Furthermore, Λ is an attractor, in the sense that there exists an open neighborhood U of Λ such that $\Lambda = \bigcap_{n\in\mathbb{N}} f^n(U)$.

Let again $f\colon M \to M$ be a C^1 diffeomorphism, and let $\Lambda \subset M$ be a hyperbolic set for f. Given $\varepsilon > 0$, for each $x \in \Lambda$ we consider the sets

$$V^s(x) = \big\{y \in M : d(f^n(y), f^n(x)) < \varepsilon \text{ for every } n \geq 0\big\},$$

and

$$V^u(x) = \big\{y \in M : d(f^n(y), f^n(x)) < \varepsilon \text{ for every } n \leq 0\big\},$$

where d is the distance in M. Provided that ε is sufficiently small, for each $x \in \Lambda$ the sets $V^s(x)$ and $V^u(x)$ are smooth manifolds containing x, such that

$$T_xV^s(x) = E^s(x) \quad \text{and} \quad T_xV^u(x) = E^u(x).$$

These are called respectively *local stable manifold* and *local unstable manifold* (of size ε) at the point x. Furthermore, there exists $\delta = \delta(\varepsilon) > 0$ such that if $x, y \in \Lambda$ with $d(x,y) < \delta$, then the intersection $V^s(x) \cap V^u(y)$ consists of a single point. The function

$$[\cdot,\cdot]\colon \big\{(x,y) \in \Lambda \times \Lambda : d(x,y) < \delta\big\} \to M$$

defined by $[x, y] = V^s(x) \cap V^u(y)$ is called a *product structure.*

Moreover, the set $\Lambda \subset M$ is said to be *locally maximal* if there is an open neighborhood U of Λ such that

$$\Lambda = \bigcap_{n\in\mathbb{Z}} f^n U.$$

For locally maximal hyperbolic sets there exists $\delta = \delta(\varepsilon) > 0$ such that if $x, y \in \Lambda$ with $d(x, y) < \delta$, then $\operatorname{card}[x, y] = 1$ and $[x, y] \in \Lambda$. In other words, we obtain a map

$$[\cdot, \cdot]\colon \big\{(x, y) \in \Lambda \times \Lambda : d(x, y) < \delta\big\} \to \Lambda. \tag{6.7}$$

A nonempty closed set $R \subset \Lambda$ is called a *rectangle* if $\operatorname{diam} R < \delta$ (with δ as in (6.7)), $R = \overline{\operatorname{int} R}$ (with the interior computed with respect to the induced topology on Λ), and $[x, y] \in R$ whenever $x, y \in R$. The notion of Markov partition of Λ (with respect to f) can now be introduced as in Definition 6.1.3 (with the interior of each set R_i computed with respect to the induced topology on Λ). Any locally maximal hyperbolic set has Markov partitions with arbitrarily small diameter (see [39]). The construction of Markov partitions is due to Sinai [184, 183] in the case of Anosov diffeomorphisms and to Bowen [35] in the general case (using the shadowing property).

6.2.2 Dimension estimates

We study in this section the dimension of hyperbolic sets for diffeomorphisms that need not be conformal. As in the case of repellers, the aim is to present the best possible estimates using the least information about the dynamics. We refer to Section 6.3 for a panorama of the existing results concerning sharper dimension estimates of hyperbolic sets when more geometric information is available.

Let Λ be a locally maximal hyperbolic set for a diffeomorphism f, and let $R_1, \ldots, R_\kappa$ be a Markov partition of Λ. For each $i = 1, \ldots, \kappa$, let $\hat{R}_i$ be a sufficiently small open neighborhood of R_i such that

$$\hat{R}_i \cap \hat{R}_j = \varnothing \quad \text{whenever} \quad R_i \cap R_j = \varnothing.$$

Let also

$$\hat{R}^u_{i_0\cdots i_n} = \bigcap_{k=0}^{n} f^{-k}\hat{R}_{i_k} \quad \text{and} \quad \hat{R}^s_{i_{-n}\cdots i_0} = \bigcap_{k=-n}^{0} f^{-k}\hat{R}_{i_k}.$$

Given a point

$$x = \chi(\cdots k_{-1}k_0k_1\cdots) \in \Lambda \cap R_{k_0},$$

we consider the sets $C^u(x)$ and $C^s(x)$ in (6.3) and (6.4). We define sequences of functions $\underline{\varphi}^u_n$ and $\overline{\varphi}^u_n$ in $C^u(x)$ by

$$\underline{\varphi}^u_n(\omega) = -\log \max\|d_y f^n|E^u\|$$

and

$$\overline{\varphi}_n^u(\omega) = \log \max \|(d_y f^n)^{-1} | E^u\|,$$

where the maximum is taken over all points $y \in V^u(x, \hat{R}^u_{i_0 \cdots i_n})$, as well as sequences of functions $\underline{\varphi}_n^s$ and $\overline{\varphi}_n^s$ in $C^s(x)$ by

$$\underline{\varphi}_n^s(\omega) = -\log \max \|d_y f^{-n} | E^s\|$$

and

$$\overline{\varphi}_n^s(\omega) = \log \max \|(d_y f^{-n})^{-1} | E^s\|,$$

where the maximum is taken over all points $y \in V^s(x, \hat{R}^u_{i_{-n} \cdots i_0})$. The four sequences have tempered variation (see (4.2)), and by the second property in Theorem 4.4.1 there exist unique roots $\underline{r}^u(x)$ and $\overline{r}^u(x)$ respectively of the equations

$$\overline{P}_{C^u(x)}(r\underline{\Phi}^u) = 0 \quad \text{and} \quad P_{C^u(x)}(r\overline{\Phi}^u) = 0,$$

and unique roots $\underline{r}^s(x)$ and $\overline{r}^s(x)$ respectively of the equations

$$\overline{P}_{C^s(x)}(r\underline{\Phi}^s) = 0 \quad \text{and} \quad P_{C^s(x)}(r\overline{\Phi}^s) = 0.$$

By analogous arguments to those in the case of repellers in (5.27)–(5.28), the following result is an immediate consequence of Theorem 6.1.4.

Theorem 6.2.4 ([5]). *Let Λ be a locally maximal hyperbolic set of a C^1 diffeomorphism which is topologically mixing on Λ. Then, for each $x \in \Lambda$, we have*

$$\underline{r}^u(x) \le \dim_H V^u(x, \Lambda) \le \underline{\dim}_B V^u(x, \Lambda) \le \overline{\dim}_B V^u(x, \Lambda) \le \overline{r}^u(x)$$

and

$$\underline{r}^s(x) \le \dim_H V^s(x, \Lambda) \le \underline{\dim}_B V^s(x, \Lambda) \le \overline{\dim}_B V^s(x, \Lambda) \le \overline{r}^s(x).$$

Now we consider a $C^{1+\alpha}$ diffeomorphism f. This means that f and f^{-1} are of class $C^{1+\alpha}$, for some $\alpha \in (0, 1]$. Let again Λ be a locally maximal hyperbolic set for f. In this case one can obtain dimension estimates for Λ using a pointwise version of the previous approach. Namely, we define sequences of functions $\underline{\varphi}_n^{*u}$ and $\overline{\varphi}_n^{*u}$ in $C^u(x)$ by

$$\underline{\varphi}_n^{*u}(\omega) = -\log \|d_y f^n | E^u\|$$

and

$$\overline{\varphi}_n^{*u}(\omega) = \log \|(d_y f^n)^{-1} | E^u\|$$

for each $\omega = (i_0 i_1 \cdots)$, where $y = \chi(\cdots k_{-2} k_{-1} i_0 i_1 \cdots)$, as well as sequences of functions $\underline{\varphi}_n^{*s}$ and $\overline{\varphi}_n^{*s}$ in $C^s(x)$ by

$$\underline{\varphi}_n^{*s}(\omega) = -\log \|d_y f^n | E^s\|$$

and

$$\overline{\varphi}_n^{*s}(\omega) = \log\|(d_y f^n)^{-1}|E^s\|$$

for each $\omega = (i_0 i_{-1} \cdots)$, where $y = \chi(\cdots i_{-1} i_0 k_1 k_2 \cdots)$. If the derivatives $df|E^u$ and $df^{-1}|E^s$ are α-bunched (see (5.29)), a slight modification of the proof of Proposition 5.2.7 shows that the four sequences satisfy property (4.2). Then, by the second property in Theorem 4.4.1, there exist unique roots $\underline{r}^{*u}(x)$ and $\overline{r}^{*u}(x)$ respectively of the equations

$$\overline{P}_{C^u(x)}(r\underline{\Phi}^{*u}) = 0 \quad \text{and} \quad P_{C^u(x)}(r\overline{\Phi}^{*u}) = 0,$$

as well as unique roots $\underline{r}^{*s}(x)$ and $\overline{r}^{*s}(x)$ respectively of the equations

$$\overline{P}_{C^s(x)}(r\underline{\Phi}^{*s}) = 0 \quad \text{and} \quad P_{C^s(x)}(r\overline{\Phi}^{*s}) = 0.$$

Moreover, one can show that

$$\underline{r}^{*u}(x) = \underline{r}^u(x), \quad \overline{r}^{*u}(x) = \overline{r}^u(x), \quad \underline{r}^{*s}(x) = \underline{r}^s(x), \quad \overline{r}^{*s}(x) = \overline{r}^s(x).$$

The following result is thus an immediate consequence of Theorem 6.2.4.

Theorem 6.2.5. *Let Λ be a locally maximal hyperbolic set of a $C^{1+\alpha}$ diffeomorphism f which is topologically mixing on Λ, such that the derivatives $df|E^u$ and $df^{-1}|E^s$ are α-bunched. Then, for each $x \in \Lambda$, we have*

$$\underline{r}^{*u}(x) \le \dim_H V^u(x,\Lambda) \le \underline{\dim}_B V^u(x,\Lambda) \le \overline{\dim}_B V^u(x,\Lambda) \le \overline{r}^{*u}(x)$$

and

$$\underline{r}^{*s}(x) \le \dim_H V^s(x,\Lambda) \le \underline{\dim}_B V^s(x,\Lambda) \le \overline{\dim}_B V^s(x,\Lambda) \le \overline{r}^{*s}(x).$$

One can also obtain dimension estimates for $V^u(x,\Lambda)$ and $V^s(x,\Lambda)$ using only the classical thermodynamic formalism, in a similar manner to that for repellers in Section 5.2. Namely, we define functions $\underline{\varphi}^u$ and $\overline{\varphi}^u$ in $C^u(x)$ by

$$\underline{\varphi}^u(\omega) = -\log\|d_y f|E^u\|$$

and

$$\overline{\varphi}^u(\omega) = \log\|(d_y f)^{-1}|E^u\|$$

for each $\omega = (i_0 i_1 \cdots)$, where $y = \chi(\cdots k_{-2} k_{-1} i_0 i_1 \cdots)$, as well as functions $\underline{\varphi}^s$ and $\overline{\varphi}^s$ in $C^s(x)$ by

$$\underline{\varphi}^s(\omega) = -\log\|d_y f^{-1}|E^s\|$$

and

$$\overline{\varphi}^s(\omega) = \log\|(d_y f^{-1})^{-1}|E^s\|$$

for each $\omega = (i_0 i_{-1} \cdots)$, where $y = \chi(\cdots i_{-1} i_0 k_1 k_2 \cdots)$. Now let $\underline{t}^u(x)$ and $\overline{t}^u(x)$ be the unique roots respectively of the equations

$$P_{C^u(x)}(t\underline{\varphi}^u) = 0 \quad \text{and} \quad P_{C^u(x)}(t\overline{\varphi}^u) = 0,$$

and let $\underline{t}^s(x)$ and $\overline{t}^s(x)$ be the unique roots respectively of the equations

$$P_{C^s(x)}(t\underline{\varphi}^s) = 0 \quad \text{and} \quad P_{C^s(x)}(t\overline{\varphi}^s) = 0.$$

One can show that

$$\underline{t}^u(x) \le \underline{r}^u(x) \le \overline{r}^u(x) \le \overline{t}^u(x)$$

and

$$\underline{t}^s(x) \le \underline{r}^s(x) \le \overline{r}^s(x) \le \overline{t}^s(x).$$

Theorem 6.2.4 thus implies the following.

Theorem 6.2.6. *Let Λ be a locally maximal hyperbolic set of a C^1 diffeomorphism which is topologically mixing on Λ. Then, for each $x \in \Lambda$, we have*

$$\underline{t}^u(x) \le \dim_H V^u(x,\Lambda) \le \underline{\dim}_B V^u(x,\Lambda) \le \overline{\dim}_B V^u(x,\Lambda) \le \overline{t}^u(x)$$

and

$$\underline{t}^s(x) \le \dim_H V^s(x,\Lambda) \le \underline{\dim}_B V^s(x,\Lambda) \le \overline{\dim}_B V^s(x,\Lambda) \le \overline{t}^s(x).$$

Incidentally, for a $C^{1+\alpha}$ diffeomorphism f with α-bunched derivatives $df|E^u$ and $df^{-1}|E^s$, we have

$$\underline{t}^u(x) \le \underline{r}^{*u}(x) \le \overline{r}^{*u}(x) \le \overline{t}^u(x)$$

and

$$\underline{t}^s(x) \le \underline{r}^{*s}(x) \le \overline{r}^{*s}(x) \le \overline{t}^s(x),$$

but all these inequalities may be strict (compare with Example 5.2.9). Therefore, the dimension estimates in Theorem 6.2.5 may be sharper than those in Theorem 6.2.6.

It is well known that the holonomy maps of a locally maximal hyperbolic set of a $C^{1+\alpha}$ diffeomorphism are Hölder continuous. Moreover, in sufficiently small rectangles the map $(x,y) \mapsto [x,y]$ is a C^β homeomorphism with C^β inverse if and only if the holonomy maps are of class C^β. This implies that

$$\gamma \sup_{x\in\Lambda}(\underline{r}^u(x) + \underline{r}^s(x)) \le \dim_H \Lambda \le \underline{\dim}_B \Lambda \le \overline{\dim}_B \Lambda \le \gamma^{-1} \sup_{x\in\Lambda}(\overline{r}^u(x) + \overline{r}^s(x)),$$

where γ is any Hölder exponent for the holonomies. For hyperbolic sets of C^2 diffeomorphisms, an effective estimate for the largest Hölder exponent of the holonomy maps was established by Schmeling and Siegmund-Schultze in [174].

6.2.3 The conformal case

We discuss in this section the particular case of hyperbolic sets of conformal diffeomorphisms, and more generally the case of hyperbolic sets of diffeomorphism that are conformal in the stable and unstable directions. In this context the dimension estimates in the former section become identities.

We first recall the notion of conformal map on a hyperbolic set.

Definition 6.2.7. We say that a diffeomorphism $f\colon M \to M$ is *conformal* on a hyperbolic set $\Lambda \subset M$ if the maps $d_x f|E^s$ and $d_x f|E^u$ are multiples of isometries for every $x \in \Lambda$.

For example, if M is a surface and $\dim E^s(x) = \dim E^u(x) = 1$ for every $x \in \Lambda$, then f is conformal on Λ.

We define functions φ^u and φ^s in Λ by

$$\varphi^u(x) = -\log\|d_x f|E^u\| \quad \text{and} \quad \varphi^s(x) = -\log\|d_x f^{-1}|E^s\|.$$

The following result is an immediate consequence of Theorem 6.2.6.

Theorem 6.2.8. *Let Λ be a locally maximal hyperbolic set of a C^1 diffeomorphism which is conformal and topologically mixing on Λ. Then, for each $x \in \Lambda$, we have*

$$\dim_H(V^s(x)\cap\Lambda) = \underline{\dim}_B(V^s(x)\cap\Lambda) = \overline{\dim}_B(V^s(x)\cap\Lambda) = t_s \tag{6.8}$$

and

$$\dim_H(V^u(x)\cap\Lambda) = \underline{\dim}_B(V^u(x)\cap\Lambda) = \overline{\dim}_B(V^u(x)\cap\Lambda) = t_u, \tag{6.9}$$

where t_s and t_u are the unique real numbers such that

$$P_\Lambda(t_s\varphi_s) = P_\Lambda(t_u\varphi_u) = 0. \tag{6.10}$$

McCluskey and Manning showed in [133] that

$$\dim_H(V^s(x)\cap\Lambda) = t_s \quad \text{and} \quad \dim_H(V^u(x)\cap\Lambda) = t_u$$

for every $x \in \Lambda$. The equality between the Hausdorff dimension and the lower and upper box dimensions is due to Takens in [187] for C^2 diffeomorphisms (see also [146]) and to Palis and Viana in [147] in the general case. Barreira [5] and Pesin [152] gave alternative proofs based on the thermodynamic formalism. We emphasize that in higher dimensions the Hausdorff and box dimensions of a locally maximal hyperbolic set may not coincide.

The Hausdorff and box dimensions of a conformal locally maximal hyperbolic set can then be computed as follows.

Theorem 6.2.9. *If Λ is a locally maximal hyperbolic set of a C^1 diffeomorphism which is conformal and topologically mixing on Λ, then*

$$\dim_H\Lambda = \underline{\dim}_B\Lambda = \overline{\dim}_B\Lambda = t_s + t_u.$$

Proof. Since the diffeomorphism is conformal, the product structure is a Lipschitz homeomorphism with Lipschitz inverse (see [86]), and thus the statement follows essentially from Theorem 6.2.8 by adding the dimensions along the stable and unstable directions. This is possible due to the equality between the Hausdorff and box dimensions in (6.8) and (6.9).

The detailed argument is as follows. Since the local product structure is a Lipschitz homeomorphism with Lipschitz inverse, the set $[V^u(x,\Lambda), V^s(x,\Lambda)]$ can be parameterized by the product $V^u(x,\Lambda) \times V^s(x,\Lambda)$ using a Lipschitz map with Lipschitz inverse. Therefore,

$$\dim_H \big[V^u(x,\Lambda), V^s(x,\Lambda)\big] = \dim_H \big(V^u(x,\Lambda) \times V^s(x,\Lambda)\big),$$

with corresponding identities for the lower and upper box dimensions. Due to the inequalities

$$\dim_H A + \dim_H B \le \dim_H (A \times B)$$

and

$$\overline{\dim}_B (A \times B) \le \overline{\dim}_B A + \overline{\dim}_B B,$$

which are valid for any subsets A and B of a metric space, we conclude that

$$\begin{aligned}\dim_H V^u(x,\Lambda) + \dim_H V^s(x,\Lambda) &\le \dim_H \big[V^u(x,\Lambda), V^s(x,\Lambda)\big] \\ &\le \underline{\dim}_B \big[V^u(x,\Lambda), V^s(x,\Lambda)\big] \\ &\le \overline{\dim}_B \big[V^u(x,\Lambda), V^s(x,\Lambda)\big] \\ &\le \underline{\dim}_B V^u(x,\Lambda) + \overline{\dim}_B V^s(x,\Lambda).\end{aligned}$$

The desired result follows now readily from Theorem 6.2.8. □

The first complete argument establishing Theorem 6.2.9 in the case of surface diffeomorphisms was given by Barreira in [5]. The proof given by Pesin in [152] includes the general case of conformal diffeomorphisms on manifolds of arbitrary dimension (the statement can also be obtained by repeating arguments in [5]). We emphasize that the proof of Theorem 6.2.9 depends crucially on the fact that f is conformal on Λ. In fact, as we have already mentioned in Section 6.2.2, any local product structure is a Hölder continuous homeomorphism with Hölder continuous inverse. But in general it is not more than Hölder continuous for a generic diffeomorphism in a certain open set, in view of work of Schmeling in [171] (see also [174]). In [196], Wolf obtained a version of Theorem 6.2.9 for Julia sets of polynomial automorphisms of $\mathbb{C}^2$.

Palis and Viana in [147] established the continuous dependence of the dimension of a hyperbolic set on the diffeomorphism. Namely, let Λ be a locally maximal hyperbolic set of a C^1 surface diffeomorphism f with $\dim E^s(x) = \dim E^u(x) = 1$ for each $x \in \Lambda$. Then there is an open neighborhood $\mathcal{U}$ of f in the space of C^1 diffeomorphisms and a continuous map $\mathcal{U} \ni g \mapsto h_g$ to the space of continuous embeddings $\Lambda_f \to M$ satisfying $h_f = \mathrm{Id}$, such that $\Lambda_g = h_g(\Lambda_f)$ is a locally maximal hyperbolic set for g, with $f|\Lambda_f$ and $g|\Lambda_g$ topologically conjugate.

Theorem 6.2.10 ([147]). *The function $g \mapsto \dim_H \Lambda_g$ is continuous.*

Mañé [129] obtained an even higher regularity for the dimension.

Theorem 6.2.11. *Let Λ be a locally maximal and totally disconnected hyperbolic set of a C^r surface diffeomorphism f, with $r \geq 2$ and $\dim E^s(x) = \dim E^u(x) = 1$ for each $x \in \Lambda$. Then there exists an open C^r-neighborhood $\mathcal{U}$ of f and a C^r map $\mathcal{U} \ni g \mapsto h_g$ to the space of continuous embeddings $\Lambda_f \to M$ such that $g \mapsto \dim_H h_g(\Lambda_f)$ is of class C^{r-1}.*

In higher-dimensional manifolds (and thus in the nonconformal case) the Hausdorff dimension of hyperbolic sets may vary discontinuously. Examples were given by Pollicott and Weiss in [158] followed by Bonatti, Díaz, and Viana in [33]. Díaz and Viana [47] studied a one-parameter family of diffeomorphisms on the 2-torus bifurcating from an Anosov map to a DA map. They showed that for an open set of these families the Hausdorff and box dimensions of the nonwandering set are discontinuous across the bifurcation.

We refer to [7] for a detailed discussion of the existence of measures of maximal dimension for conformal hyperbolic sets. These measures are a dimensional counterpart of the measures of maximal entropy. A crucial difference is that while the entropy is upper semicontinuous (in this setting), the Hausdorff dimension is never upper semicontinuous. The existence of ergodic invariant measures of maximal dimension was established by Barreira and Wolf in [25] for diffeomorphisms on surfaces with one-dimensional stable and unstable distributions, although their approach generalizes without change to the more general case of conformal hyperbolic sets. See [197] for a related result of Wolf in the particular case of polynomial automorphisms of $\mathbb{C}^2$. It was shown by Rams in [161] that in general it does not exist a unique ergodic invariant measure of maximal dimension, even in the case of linear horseshoes (more precisely, he showed that there is a one-parameter family of Bernoulli measures of maximal dimension).

6.3 Further developments

We give in this section a panorama of the existing results concerning the dimension of a hyperbolic set for a diffeomorphism when more geometric information is available.

We first describe a result of Franz in [71]. Let $\Lambda \subset M$ be a compact f-invariant set for a C^1 diffeomorphism f with an equivariant splitting $T_\Lambda M = E^1 \oplus E^2$. This means that

$$d_x f E^i(x) = E^i(f(x)), \quad i = 1, 2$$

for every $x \in \Lambda$. We note that the set Λ need not be hyperbolic. We relabel the singular values

$$\sigma_1(d_x f|E^1), \ldots, \sigma_{\dim E^1}(d_x f|E^1), \sigma_1((d_y f|E^2)^{-1}), \ldots, \sigma_{\dim E^2}((d_y f|E^2)^{-1})$$

as $\sigma_1(x,y) \geq \cdots \geq \sigma_{\dim M}(x,y)$, and for each $s \in [0, \dim M]$ we define

$$\overline{\omega}_s(x,y) = \sigma_1(x,y) \cdots \sigma_{\lfloor s \rfloor}(x,y) \sigma_{\lfloor s \rfloor + 1}(x,y)^{s - \lfloor s \rfloor}.$$

Franz showed in [71] that

$$\dim_H \Lambda \le \inf \left\{ s \in [0, \dim M] \colon \log \max_{x,y \in \Lambda} \overline{\omega}_s(x,y) < -2h(f|\Lambda) \right\},$$

where $h(f|\Lambda)$ denotes the topological entropy of $f|\Lambda$. This result generalizes work of Gu in [81] (building on former work of Fathi in [62]) for hyperbolic sets of C^2 maps satisfying a certain pinching condition.

The following result of Gelfert in [75] assumes the existence of an equivariant subbundle with respect to which f is volume-expanding.

Theorem 6.3.1. *If Λ is a compact f-invariant set of an expansive C^1 local diffeomorphism f, and $E \subset T_\Lambda M$ is an equivariant subbundle, then*

$$\overline{\dim}_B \Lambda \le \operatorname{codim} E + \inf \Big\{ s \in [0, \operatorname{rank} E] \colon \max_{x \in \Lambda} \omega_s((d_x f|E)^{-1}) < 1 \text{ and } P_\Lambda\big(\log \omega_s((d_x f|E)^{-1})\big) < 0 \Big\},$$

with ω_s as in (5.36)*, and with the convention that* $\inf \varnothing = \operatorname{rank} E$*.*

Shafikov and Wolf showed in [176] that if Λ is a hyperbolic set for a C^2 diffeomorphism $f \colon M \to M$, then

$$\dim_H \Lambda \le \dim M + \frac{P_\Lambda(-\log|\det df|E^u|)}{\lambda}, \tag{6.11}$$

where

$$\lambda = \lim_{n \to \infty} \frac{1}{n} \log \max_{x \in \Lambda} \|d_x f^n\|.$$

We note that inequality (6.11) follows from Theorem 6.3.1. We refer to [191] for the case of endomorphisms.

Related ideas to those in the proof of Theorem 5.3.5 were applied by Simon and Solomyak in [180] to compute the Hausdorff dimension of a class of horseshoes in $\mathbb{R}^3$, obtained from a $C^{1+\alpha}$ transformation of the form

$$f(x,y,z) = (\varphi(x,z) + a_k, \psi(y,z) + b_k, \zeta(z)), \tag{6.12}$$

with

$$|\partial\varphi/\partial x|, |\partial\psi/\partial y| < 1/2 \quad \text{and} \quad |\zeta'| > 1.$$

More precisely, we assume that there are disjoint closed intervals $I_1, \ldots, I_\kappa$ whose union is a proper subset of $[0,1]$ such that letting $\Delta_i = [0,1]^2 \times I_i$ we have

$$f\left(\bigcup_{i=1}^{\kappa} \Delta_i \right) \subset (0,1)^2 \times [0,1],$$

with f given by (6.12) in Δ_i. Then the limit set

$$\Lambda = \bigcap_{n\in\mathbb{Z}} f^n([0,1]^3)$$

is a horseshoe for f. Now let s_1, s_2, r_1, and r_2 be the unique roots of the equations

$$P_\Lambda\Big(s_1 \log\Big|\frac{\partial\varphi}{\partial x}\Big|\Big) = 0, \quad P_\Lambda\Big(s_2 \log\Big|\frac{\partial\psi}{\partial y}\Big|\Big) = 0,$$

and

$$P_\Lambda\Big(\log\Big(\Big|\frac{\partial\varphi}{\partial x}\Big|\cdot\Big|\frac{\partial\psi}{\partial y}\Big|^{r_1}\Big)\Big) = 0, \quad P_\Lambda\Big(\log\Big(\Big|\frac{\partial\psi}{\partial y}\Big|\cdot\Big|\frac{\partial\varphi}{\partial x}\Big|^{r_2}\Big)\Big) = 0.$$

We also set

$$s = \max\{s_1, s_2\} \quad \text{and} \quad r = \max\{r_1, r_2\}.$$

Finally, let t be the unique root of the equation $P_\Lambda(-t\log|\zeta'|) = 0$.

Theorem 6.3.2 ([180]). *For Lebesgue almost every vector* $(a_1, b_1, \ldots, a_\kappa, b_\kappa)$*:*

(i) *if* $s \le 1$*, then*

$$\dim_H \Lambda = \underline{\dim}_B \Lambda = \overline{\dim}_B \Lambda = t + s;$$

(ii) *if* $s > 1$*, then*

$$\dim_H \Lambda = \underline{\dim}_B \Lambda = \overline{\dim}_B \Lambda = t + 1 + \min\{1, r\}.$$

We emphasize that the Hausdorff and box dimensions of Λ do not depend on the vector $(a_1, b_1, \ldots, a_\kappa, b_\kappa)$.

Finally, we consider more general solenoids than those in Example 6.2.3. We recall that a *solenoid* is a hyperbolic set of the form

$$\Lambda = \bigcap_{n=1}^{\infty} f^n T,$$

where $T \subset \mathbb{R}^3$ is diffeomorphic to a solid torus $S^1 \times D$, for some closed disk $D \subset \mathbb{R}^2$, and where the map $f\colon T \to T$ extends to a diffeomorphism in some open neighborhood of T such that for each $x \in S^1$ the *section*

$$\Lambda_x = f(T) \cap (\{x\} \times D)$$

is a disjoint union of a fixed number of sets homeomorphic to a closed disk. Bothe [34] and then Simon [179] (also using his methods in [178] for noninvertible transformations) studied the dimension of solenoids (see [152, 175] for a related discussion). In particular, it is shown in [34] that under certain conditions on the diffeomorphism the map $x \mapsto \dim_H \Lambda_x$ is constant (even though the holonomies are typically not Lipschitz). More recently, Hasselblatt and Schmeling conjectured in [88] (see also [87]) that, in spite of the difficulties due to the possible low regularity of the holonomies, the Hausdorff dimension of a hyperbolic set can be computed by adding the dimensions of the stable and unstable sections. They proved this conjecture for a class of solenoids, by showing that the Hausdorff dimension of the sections is in fact constant.

We also mention some related results in the case of nonuniformly hyperbolic invariant sets. Namely, Hirayama [89] gave an upper bound for the Hausdorff dimension of the stable set of the set of typical points for a hyperbolic measure, Fan, Jiang and Wu [61] studied the dimension of the maximal invariant set of an asymptotically nonhyperbolic family, and Urbánski and Wolf [192] considered horseshoe maps that are uniformly hyperbolic except at a parabolic point, in particular establishing a dimension formula for the horseshoe.

Part III

Subadditive Thermodynamic Formalism

Chapter 7

Asymptotically Subadditive Sequences

We have already considered earlier the class of subadditive sequences (see Definition 4.2.5), and we gave several alternative formulas for the topological pressure. We consider in this chapter the more general class of asymptotically subadditive sequences, and we develop the theory in several directions. In particular, we present a variational principle for the topological pressure of an arbitrary asymptotically subadditive sequence. The proof can be described as an elaboration of the proof of the classical variational principle in Theorem 2.3.1, although it requires a special care in order to consider the more general class of asymptotically subadditive sequences. We also revisit the problem of giving alternative formulas for the topological pressure, now also taking advantage of the variational principle. Finally, we establish the existence of equilibrium measures for an arbitrary asymptotically subadditive sequence, for maps with upper semicontinuous entropy, and we consider briefly the particular case of symbolic dynamics.

7.1 Asymptotically subadditive sequences

We introduce in this section the notions of an asymptotically subadditive sequence and of an asymptotically additive sequence. We also give several examples.

Let $f\colon X \to X$ be a continuous transformation of a compact metric space.

Definition 7.1.1. A sequence of functions $\Phi = (\varphi_n)_{n\in\mathbb{N}}$ is said to be *asymptotically subadditive* if for each $\varepsilon > 0$ there exists a subadditive sequence $(\psi_n)_{n\in\mathbb{N}}$ such that

$$\limsup_{n\to\infty} \frac{\|\varphi_n - \psi_n\|_\infty}{n} < \varepsilon, \tag{7.1}$$

where $\|\varphi\|_\infty = \sup\{|\varphi(x)| : x \in X\}$.

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Assuming that Φ is an asymptotically subadditive sequence with tempered variation (see (4.2)) one can compute its nonadditive topological pressure (see Theorem 4.1.2). Here we present an alternative definition given by Feng and Huang in [66] (following a corresponding approach in [42]), for an arbitrary asymptotically subadditive sequence (see also the appendix of [5] for the description of several related notions).

Definition 7.1.2. The *topological pressure* of an asymptotically subadditive sequence Φ (with respect to f) is defined by

$$P(\Phi) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \sup_E \sum_{x\in E} \exp \varphi_n(x), \tag{7.2}$$

where the supremum is taken over all (n, ε)-separated sets $E \subset X$.

We note that (7.2) mimics the definition of the classical topological pressure in (2.1) in terms of separated sets. In the particular case of a subadditive sequence Φ with tempered variation, it follows from Theorems 4.2.6 and 4.2.7 that $P(\Phi)$ coincides with the nonadditive topological pressure $P_X(\Phi)$. This result was also established in [42, Proposition 4.7].

We also consider the particular case of asymptotically additive sequences.

Definition 7.1.3. A sequence of functions $\Phi = (\varphi_n)_{n\in\mathbb{N}}$ is said to be *asymptotically additive* if for each $\varepsilon > 0$ there exists an additive sequence $(\psi_n)_{n\in\mathbb{N}}$ for which property (7.1) holds.

The notions introduced in Definitions 7.1.1 and 7.1.3 are slight rewritings of corresponding notions introduced by Feng and Huang in [66].

It is also convenient to have the following characterization of an asymptotically additive sequence.

Proposition 7.1.4 ([66]). *A sequence Φ is asymptotically additive if and only if Φ and $-\Phi = (-\varphi_n)_{n\in\mathbb{N}}$ are asymptotically subadditive sequences.*

Proof. We first assume that Φ is asymptotically additive. Then clearly Φ is asymptotically subadditive, since the additive sequence $(\psi_n)_{n\in\mathbb{N}}$ in (7.1) is also subadditive. To verify that $-\Phi$ is asymptotically subadditive, we note that $(-\psi_n)_{n\in\mathbb{N}}$ is also additive, and

$$\frac{|-\varphi_n(x) - (-\psi_n(x))|}{n} = \frac{|\varphi_n(x) - \psi_n(x)|}{n}.$$

Now we assume that Φ and $-\Phi$ are asymptotically subadditive. We claim that for each $\varepsilon > 0$, there exist $K > 0$ and $C_{\varepsilon,k} > 0$ for each $k \geq K$ such that

$$\left|\varphi_n(x) - \frac{1}{k}\sum_{j=0}^{n-1} \varphi_k(f^j(x))\right| \leq n\varepsilon + C_{\varepsilon,k} \tag{7.3}$$

for every $n \geq 2k$ and $x \in X$. This implies that for the additive sequence $(\psi_n)_{n\in\mathbb{N}}$ obtained from the function $\psi_1 = \varphi_k/k$ we have

$$\frac{|\varphi_n(x) - \psi_n(x)|}{n} = \frac{1}{n}\left|\varphi_n(x) - \frac{1}{k}\sum_{j=0}^{n-1}\varphi_k(f^j(x))\right| \leq \varepsilon + \frac{C_{\varepsilon,k}}{n}$$

for every $n \geq 2k$ and $x \in X$, and hence, the sequence Φ is asymptotically additive.

To establish inequality (7.3), we note that since Φ is asymptotically subadditive, given $\varepsilon > 0$ there exist a subadditive sequence $(\psi_n)_{n\in\mathbb{N}}$ and $K > 0$ such that

$$|\varphi_n(x) - \psi_n(x)| \leq \frac{n\varepsilon}{2} \tag{7.4}$$

for every $n \geq K$ and $x \in X$. Set $C = \max\left\{0, \sup_{x\in X}\psi_1(x)\right\}$.

Lemma 7.1.5. *For each $n \geq 2k$ and $x \in X$ we have*

$$k\psi_n(x) \leq 2k^2C + \sum_{j=0}^{n-k}\psi_k(f^j(x)).$$

Proof of the lemma. For each $j = 0, 1, \ldots, k-1$ we have

$$\psi_n(x) \leq \psi_j(x) + \psi_{n-j}(f^j(x)) \leq jC + \psi_{n-j}(f^j(x)).$$

Hence,

$$\begin{aligned} k\psi_n(x) &\leq \sum_{j=0}^{k-1}\left(jC + \psi_{n-j}(f^j(x))\right) \\ &\leq k^2C + \sum_{j=0}^{k-1}\psi_{n-j}(f^j(x)). \end{aligned} \tag{7.5}$$

Let us observe that for each j between 0 and $k-1$, we have

$$\begin{aligned} \psi_{n-j}(f^j(x)) &\leq \sum_{l=0}^{t_j-1}\psi_k(f^{kl+j}(x)) + \psi_{n-j-kt_j}(f^{kt_j+j}(x)) \\ &\leq kC + \sum_{l=0}^{t_j-1}\psi_k(f^{kl+j}(x)), \end{aligned}$$

where t_j is the largest integer t such that $kt + j \leq n$. Together with (7.5) this implies that

$$\begin{aligned} k\psi_n(x) &\leq 2k^2C + \sum_{j=0}^{k-1}\sum_{l=0}^{t_j-1}\psi_k(f^{kl+j}(x)) \\ &= 2k^2C + \sum_{j=0}^{n-k}\psi_k(f^j(x)). \end{aligned}$$

This yields the desired inequality. □

It follows from the lemma that

$$\psi_n(x) \le 2kC + \frac{1}{k}\sum_{j=0}^{n-k} \psi_k(f^j(x))$$

for every $n \ge 2k$ and $x \in X$. Combining this with (7.4), for each $k \ge K$ we obtain

$$\begin{aligned}\psi_n(x) &\le \frac{(2n-k)\varepsilon}{2} + 2kC + \frac{1}{k}\sum_{i=0}^{n-k} \varphi_k(f^i(x)) \\ &\le \frac{(2n-k)\varepsilon}{2} + 2kC + M_k + \frac{1}{k}\sum_{i=0}^{n-1} \varphi_k(f^i(x))\end{aligned}$$

for every $n \ge 2k$ and $x \in X$, where $M_k = \max\{1, \|\varphi_k\|_\infty\}$. This shows that

$$\varphi_n(x) \le \frac{1}{k}\sum_{j=0}^{n-1} \varphi_k(f^j(x)) + n\varepsilon + C_{\varepsilon,k} \tag{7.6}$$

with $C_{\varepsilon,k} = 2kC + M_k$. Since $-\Phi$ is asymptotically subadditive, an analogous argument establishes the corresponding inequality for this sequence, that is,

$$\varphi_n(x) \ge \frac{1}{k}\sum_{j=0}^{n-1} \varphi_k(f^j(x)) - n\varepsilon - C_{\varepsilon,k}.$$

Together with (7.6) this establishes (7.3), and thus, the sequence Φ is asymptotically additive. □

Clearly, any subadditive sequence is asymptotically subadditive and any additive sequence is asymptotically additive. The following result describes some more elaborate examples.

Proposition 7.1.6 ([66]). *For a sequence of continuous functions $\Phi = (\varphi_n)_{n\in\mathbb{N}}$:*

1. *if there exists a constant $C > 0$ such that*

$$\varphi_{n+m}(x) \le C + \varphi_n(x) + \varphi_m(f^n(x))$$

for every $n, m \in \mathbb{N}$ and $x \in X$, then Φ is asymptotically subadditive;

2. *if there exists a constant $C > 0$ such that*

$$-C + \varphi_n(x) + \varphi_m(f^n(x)) \le \varphi_{n+m}(x) \le C + \varphi_n(x) + \varphi_m(f^n(x))$$

for every $n, m \in \mathbb{N}$ and $x \in X$, then Φ is asymptotically additive;

3. *if there exists a continuous function* $\varphi\colon X \to \mathbb{R}$ *such that*

$$\varphi_{n+1} - \varphi_n \circ f \to \varphi \text{ uniformly on } X \tag{7.7}$$

when $n \to \infty$*, then* Φ *is asymptotically additive.*

Proof. For the first property, we define $\psi_n = \varphi_n + C$. Then

$$\psi_{n+m}(x) \le \psi_n(x) + \psi_m(f^n(x)),$$

and the sequence $(\psi_n)_{n\in\mathbb{N}}$ is subadditive. Since

$$\frac{\varphi_n(x) - \psi_n(x)}{n} = \frac{C}{n} \to 0 \quad \text{when} \quad n \to \infty,$$

the sequence Φ is asymptotically subadditive. The second property follows readily from the first property together with the characterization of an asymptotically additive sequence in Proposition 7.1.4.

Finally, for the third property, we set

$$r_n = \sup_{x\in X} |\varphi_n(x) - \varphi_{n-1}(f(x)) - \varphi(x)|.$$

By hypothesis, $r_n \to 0$ when $n \to \infty$. For the additive sequence $\psi_n = \sum_{i=0}^{n-1} \varphi \circ f^i$ we have

$$\begin{aligned}|\varphi_n(x) - \psi_n(x)| &\le \left|\sum_{i=1}^{n} [\varphi(f^{n-i}(x)) - \varphi_{i-1}(f^{n-i+1}(x)) - \varphi(f^{n-i}(x))]\right| \\ &\le \sum_{i=1}^{n} \bigl|\varphi_i(f^{n-i}(x)) - \varphi_{i-1}(f^{n-i+1}(x)) - \varphi(f^{n-i}(x))\bigr| \le \sum_{i=1}^{n} r_i.\end{aligned}$$

Therefore,

$$\limsup_{n\to\infty} \frac{1}{n} \sup_{x\in X} |\varphi_n(x) - \psi_n(x)| \le \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} r_i = 0,$$

and the sequence Φ is asymptotically additive. □

We note that condition (7.7) occurs in Theorem 4.3.1 (in the particular case when $Z = X$).

7.2 Variational principle

We establish in this section a variational principle for the topological pressure of a subadditive sequence obtained by Cao, Feng and Huang in [42]. The proof can be described as an elaboration of the proof of the classical variational principle in

Theorem 2.3.1, although it requires a special care in order to consider arbitrary asymptotically subadditive sequences.

In fact, the variational principle is a particular case of the following variational principle for an arbitrary asymptotically subadditive sequence, obtained by Feng and Huang in [66] after a minor modification of the proof in [42].

Theorem 7.2.1 (Variational principle). *Let $f\colon X \to X$ be a continuous transformation of a compact metric space with $h(f) < \infty$. If Φ is an asymptotically subadditive sequence of continuous functions, then*

$$P(\Phi) = \sup_{\mu}\left(h_\mu(f) + \lim_{n\to\infty}\frac{1}{n}\int_X \varphi_n\,d\mu\right), \tag{7.8}$$

where the supremum is taken over all f-invariant probability measures in X.

Proof. As we already mentioned, the proof can be described as an elaboration of the proof of Theorem 2.3.1.

We first show that the limit $\lim_{n\to\infty}\int_X(\varphi_n/n)\,d\mu$ in (7.8) is well defined. Given $\varepsilon > 0$, there is a subadditive sequence $(\psi_n)_{n\in\mathbb{N}}$ such that $\|\varphi_n - \psi_n\|_\infty < n\varepsilon$ for all sufficiently large n. Moreover, since

$$\int_X \psi_{n+m}\,d\mu \le \int_X (\psi_n + \psi_m \circ f^n)\,d\mu = \int_X \psi_n\,d\mu + \int_X \psi_m\,d\mu,$$

the sequence $\int_X(\psi_n/n)\,d\mu$ converges. Therefore,

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{n}\int_X \varphi_n\,d\mu &\le \lim_{n\to\infty}\frac{1}{n}\int_X \psi_n\,d\mu + \varepsilon\\ &\le \liminf_{n\to\infty}\frac{1}{n}\int_X \varphi_n\,d\mu + 2\varepsilon.\end{aligned}$$

Since ε is arbitrary, this yields the existence of the limit in (7.8).

As in the proof of Theorem 2.3.1, we first obtain a lower bound for the topological pressure. Again, let $\eta = \{C_1, \ldots, C_k\}$ be a measurable partition of X, and given $\delta > 0$, for each $i = 1, \ldots, k$, let $D_i \subset C_i$ be a compact set such that $\mu(C_i \setminus D_i) < \delta$. Then for the measurable partition

$$\beta = \{D_0, D_1, \ldots, D_k\}, \quad \text{where} \quad D_0 = X \setminus \bigcup_{i=1}^k D_i,$$

we have $H_\mu(\eta|\beta) < 1$, provided that δ is sufficiently small (see Lemma 2.3.2). We also define

$$\Delta = \inf\bigl\{d(x,y) : x \in D_i, y \in D_j, i \ne j\bigr\},$$

and we take $\varepsilon \in (0, \Delta/2)$.

Given $n \in \mathbb{N}$, for each $C \in \beta_n := \bigvee_{j=0}^{n-1} f^{-j}\beta$ there exists $x_C \in \overline{C}$ such that

$$\varphi_n(x_C) = \sup\bigl\{\varphi_n(x) : x \in C\bigr\}.$$

We claim that for each $C \in \beta_n$ there are at most 2^n sets $C' \in \beta_n$ such that $d_n(x_C, x_{C'}) < \varepsilon$. For the proof, let $i_0(C), i_1(C), \ldots, i_{n-1}(C) \in \{0, 1, \ldots, k\}$ be the unique numbers such that

$$C = \bigcap_{j=0}^{n-1} f^{-j} D_{i_j(C)}.$$

Let also $Y = Y(C)$ be the collection of all sets $C' \in \beta_n$ such that $d_x(x_C, x_{C'}) < \varepsilon$. Then

$$\operatorname{card}\big\{i_l(C') : C' \in Y\big\} \leq 2 \quad \text{for} \quad l = 0, 1, \ldots, n-1. \tag{7.9}$$

Indeed, let us assume on the contrary that there exist $l \in \{0, \ldots, n-1\}$ and $C_1', C_2', C_3' \in Y$ such that $i_l(C_1'), i_l(C_2')$ and $i_l(C_3')$ are distinct. Without loss of generality we assume that $i_l(C_1') \neq 0$ and $i_l(C_2') \neq 0$. Then

$$\begin{aligned} d_n(x_{C_1'}, x_{C_2'}) &\geq d\big(f^l(x_{C_1'}), f^l(x_{C_2'})\big) \geq d\big(D_{i_l(C_1')}, D_{i_l(C_2')}\big) \\ &\geq \Delta > 2\varepsilon \geq d_n(x_C, x_{C_1'}) + d_n(x_C, x_{C_2'}). \end{aligned}$$

This contradicts the triangle inequality, which establishes property (7.9).

The following step is the construction of an (n, ε)-separated set E with respect to f, such that

$$2^n \sum_{x \in E} \exp \varphi_n(x) \geq \sum_{C \in \beta_n} \exp \varphi_n(x_C). \tag{7.10}$$

We first take $F_1 \in \beta_n$ such that

$$\varphi_n(x_{F_1}) = \max_{C \in \beta_n} \varphi_n(x_C).$$

Let also $Z_1 = Y(F_1)$. By (7.9), we have $\operatorname{card} Z_1 \leq 2^n$. If the collection $\beta_n \setminus Z_1$ is nonempty, we take $F_2 \in \beta_n \setminus Z_1$ such that

$$\varphi_n(x_{F_2}) = \max_{C \in \beta_n \setminus Z_1} \varphi_n(x_C).$$

Now let Z_2 be the collection of all sets $C' \in \beta_n \setminus Z_1$ such that $d_n(x_{F_2}, x_{C'}) < \varepsilon$, and let us restart the process inductively. Namely, at step m we take

$$F_m \in \beta_n \setminus \bigcup_{j=1}^{m-1} Z_j$$

such that

$$\varphi_n(x_{F_m}) = \max\left\{\varphi_n(x_C) : C \in \beta_n \setminus \bigcup_{j=1}^{m-1} Z_j\right\}.$$

Since the partition β_n is finite, the process ends at some step m. Set

$$E = \big\{x_{F_j} : j = 1, \ldots, m\big\}.$$

Then E is an (n,ε)-separated set, and

$$\begin{aligned}\sum_{x\in E}\exp\varphi_n(x) &= \sum_{j=1}^m \exp\varphi_n(x_{F_j})\\ &\ge \sum_{j=1}^m 2^{-n}\sum_{C\in Z_j}\exp\varphi_n(x_C)\\ &= 2^{-n}\sum_{C\in\beta_n}\exp\varphi_n(x_C),\end{aligned}$$

which yields (7.10). Proceeding in a similar manner to that in (2.34) and (2.35), for each f-invariant probability measure μ in X we obtain

$$\begin{aligned}\frac{1}{n}H_\mu(\beta_n)+\frac{1}{n}\int_X\varphi_n\,d\mu &\le \frac{1}{n}\sum_{C\in\beta_n}\mu(C)(-\log\mu(C)+\varphi_n(x_C))\\ &\le \frac{1}{n}\log\sum_{C\in\beta_n}\exp\varphi_n(x_C)\\ &\le \frac{1}{n}\log\left(2^n\sum_{x\in E}\exp\varphi_n(x)\right)\\ &\le \log 2+\frac{1}{n}\log\sum_{x\in E}\exp\varphi_n(x).\end{aligned}$$

In a similar manner to that in (2.36), letting $n\to\infty$ yields

$$\begin{aligned}h_\mu(f,\eta)+\lim_{n\to\infty}\frac{1}{n}\int_X\varphi_n\,d\mu &\le h_\mu(f,\beta)+H_\mu(\eta|\beta)+\lim_{n\to\infty}\frac{1}{n}\int_X\varphi_n\,d\mu\\ &\le 1+\log 2+\limsup_{n\to\infty}\frac{1}{n}\log\sum_{x\in E}\exp\varphi_n(x).\end{aligned}$$

Hence, letting $\varepsilon\to 0$, we obtain

$$h_\mu(f)+\lim_{n\to\infty}\frac{1}{n}\int_X\varphi_n\,d\mu\le 1+\log 2+P_f(\varphi). \tag{7.11}$$

In order to proceed we establish an auxiliary result.

Lemma 7.2.2. *For each $k\in\mathbb{N}$ we have $P_{f^k}(\Phi_k)=kP(\Phi)$, where $P_{f^k}(\Phi_k)$ is the topological pressure of the (asymptotically subadditive) sequence $\Phi_k=(\varphi_{kn})_{n\in\mathbb{N}}$ with respect to f^k.*

Proof of the lemma. We note that if E is an (n,ε)-separated set with respect to f^k, then it is also an (nk,ε)-separated set with respect to f. Therefore,

$$\sup_E\sum_{x\in E}\exp\varphi_{kn}(x)\le\sup_F\sum_{x\in F}\exp\varphi_{kn}(x),$$

where E is any (n,ε)-separated set with respect to f^k, and where F is any (nk,ε)-separated set with respect to f. This implies that

$$P_{f^k}(\Phi_k) \le \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log\sup_F\sum_{x\in F}\exp\varphi_{kn}(x) \le kP(\Phi).$$

For the reverse inequality, given $\alpha>0$, let $(\psi_n)_{n\in\mathbb{N}}$ be a subadditive sequence such that

$$\limsup_{n\to\infty}\frac{\|\varphi_n-\psi_n\|_\infty}{n}<\alpha.$$

Moreover, for each $\varepsilon>0$, let us take $\delta>0$ such that $d_k(x,y)<\varepsilon$ whenever $d(x,y)<\delta$. Given $n\in\mathbb{N}$, let now l be an arbitrary integer in $[kn,k(n+1))$. It follows from the choice of δ that any (l,ε)-separated set with respect to f is also an (n,δ)-separated set with respect to f^k. Moreover, for any sufficiently large n we have

$$\begin{aligned}\varphi_l(x) &\le \psi_l(x)+l\alpha\\ &\le \psi_{nk}(x)+\psi_{l-nk}(f^{nk}(x))+l\alpha\\ &\le \varphi_{nk}(x)+kC+2l\alpha,\end{aligned}$$

where $C=\max\{0,\sup_{x\in X}\varphi_1\}$. Therefore,

$$\sup_E\sum_{x\in E}\exp\varphi_l(x)\le e^{kC+2l\alpha}\sup_F\sum_{x\in F}\exp\varphi_{nk}(x),$$

where E is any (l,ε)-separated set with respect to f, and where F is any (n,δ)-separated set with respect to f^k. This implies that

$$\begin{aligned}P(\Phi) &\le \lim_{\delta\to 0}\limsup_{l\to\infty}\frac{1}{l}\log\left(e^{kC+2l\alpha}\sup_F\sum_{x\in F}\exp\varphi_{nk}(x)\right)\\ &= 2\alpha+P_{f^k}(\Phi_k).\end{aligned}$$

Since α is arbitrary, we conclude that $P(\Phi)\le P_{f^k}(\Phi_k)$. □

We proceed with the proof of the theorem. Since

$$h_\mu(f^k)=kh_\mu(f)\quad\text{for}\quad k\in\mathbb{N},$$

it follows from (7.11) and Lemma 7.2.2 that

$$\begin{aligned}h_\mu(f)+\lim_{n\to\infty}\frac{1}{n}\int_X\varphi_n\,d\mu &= \frac{1}{k}\biggl(h_\mu(f^k)+\lim_{n\to\infty}\frac{1}{n}\int\varphi_{kn}\,d\mu\biggr)\\ &\le \frac{1}{k}\bigl(1+\log 2+P_{f^k}(\Phi_k)\bigr)\\ &= \frac{1+\log 2}{k}+P(\Phi).\end{aligned}$$

Letting $k \to \infty$ yields

$$h_\mu(f) + \lim_{n\to\infty} \frac{1}{n} \int_X \varphi_n \, d\mu \le P(\Phi),$$

and hence,

$$\sup_\mu \left(h_\mu(f) + \lim_{n\to\infty} \frac{1}{n} \int_X \varphi_n \, d\mu \right) \le P(\Phi).$$

Now we establish the reverse inequality. Given $\varepsilon > 0$ and $n \in \mathbb{N}$, let E_n be an (n, ε)-separated set (with respect to f) satisfying (2.38). We also consider the measures ν_n and μ_n respectively in (2.39) and (2.40). Finally, let $(k_n)_{n\in\mathbb{N}} \subset \mathbb{N}$ be a sequence satisfying (2.41), such that the sequence of measures $(\mu_{k_n})_{n\in\mathbb{N}}$ converges to some measure μ. We note that μ is an f-invariant probability measure in X. For the partition ξ constructed in the proof of Theorem 2.3.1, we can repeat corresponding arguments in the proof (see (2.42) and (2.45)) to obtain

$$H_{\nu_n}(\xi_n) + \int_X \varphi_n \, d\nu_n = \log \sum_{x\in E_n} \exp \varphi_n(x),$$

and thus also

$$\frac{1}{n} \log \sum_{x\in E_n} \exp \varphi_n(x) \le \frac{1}{m} H_{\mu_n}(\xi_m) + \frac{2m}{n} \log \operatorname{card} \xi + \frac{1}{n} \int_X \varphi_n \, d\mu_n,$$

where $\xi_n = \bigvee_{j=0}^{n-1} f^{-j}\xi$. It follows from the choice of $(k_n)_{n\in\mathbb{N}}$ in (2.41) that

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{n} \log \sum_{x\in E_n} \exp \varphi_n(x) \\
&= \lim_{n\to\infty} \frac{1}{k_n} \log \sum_{x\in E_{k_n}} \exp \varphi_{k_n}(x) \\
&\le \lim_{n\to\infty} \left(\frac{1}{m} H_{\mu_{k_n}}(\xi_m) + \frac{2m}{k_n} \log \operatorname{card} \xi + \frac{1}{k_n} \int_X \varphi_{k_n} \, d\mu_{k_n} \right) \\
&= \frac{1}{m} H_\mu(\xi_m) + \limsup_{n\to\infty} \frac{1}{k_n} \int_X \varphi_{k_n} \, d\mu.
\end{aligned} \tag{7.12}$$

We also need the following auxiliary statement.

Lemma 7.2.3. *We have*

$$\limsup_{n\to\infty} \frac{1}{k_n} \int_X \varphi_{k_n} \, d\nu_{k_n} \le \lim_{n\to\infty} \frac{1}{n} \int_X \varphi_n \, d\mu.$$

Proof of the lemma. We first consider a subadditive sequence $(\psi_n)_{n\in\mathbb{N}}$. For each $n \geq 2k$, it follows from Lemma 7.1.5 that

$$\begin{aligned}\frac{1}{n}\int_X \psi_n\,d\nu_n &= \frac{1}{kn}\int_X k\psi_n\,d\nu_n \\ &\leq \frac{1}{kn}\left(2k^2C + \int_X \sum_{j=0}^{n-k} \psi_k(f^j(x))\,d\nu_n(x)\right) \\ &= \frac{2kC}{n} + \frac{n-k+1}{kn}\int_X \psi_k\,d\bar{\mu}_{n,k},\end{aligned} \tag{7.13}$$

where

$$\bar{\mu}_{n,k} = \frac{1}{n-k+1}\sum_{j=0}^{n-k} f_*^j \nu_n.$$

Let us observe that for any continuous function $\varphi\colon X \to \mathbb{R}$, we have

$$\begin{aligned}\left|n\int_X \varphi\,d\mu_n - (n-k+1)\int_X \varphi\,d\bar{\mu}_{n,k}\right| &= \left|\sum_{i=n-k+1}^{n-1}\int_X \varphi(f^i(x))\,d\nu_n(x)\right| \\ &\leq (k-1)\|\varphi\|_\infty.\end{aligned}$$

This yields

$$\lim_{n\to\infty}\int_X \varphi\,d\bar{\mu}_{k_n,k} = \lim_{n\to\infty}\int_X \varphi\,d\mu_{k_n} = \int_X d\mu.$$

It thus follows from (7.13) that

$$\limsup_{n\to\infty}\frac{1}{k_n}\int_X \psi_{k_n}\,d\nu_{k_n} \leq \frac{1}{k}\limsup_{n\to\infty}\int_X \psi_k\,d\bar{\mu}_{k_n,k} = \frac{1}{k}\int_X \psi_k\,d\mu,$$

and hence,

$$\limsup_{n\to\infty}\frac{1}{k_n}\int_X \varphi_{k_n}\,d\nu_{k_n} \leq \lim_{k\to\infty}\frac{1}{k}\int_X \psi_k\,d\mu.$$

Now let Φ be an asymptotically subadditive sequence. Given $\varepsilon > 0$, there exists a subadditive sequence $(\psi_n)_{n\in\mathbb{N}}$ such that $\|\varphi_n - \psi_n\|_\infty < n\varepsilon$ for all sufficiently large n. Therefore,

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{k_n}\int_X \varphi_{k_n}\,d\nu_{k_n} &\leq \limsup_{n\to\infty}\frac{1}{k_n}\int_X \psi_{k_n}\,d\nu_{k_n} + \varepsilon \\ &\leq \lim_{n\to\infty}\frac{1}{n}\int_X \psi_n\,d\mu + \varepsilon \\ &\leq \lim_{n\to\infty}\frac{1}{n}\int_X \varphi_n\,d\mu + 2\varepsilon.\end{aligned}$$

Since ε is arbitrary, we obtain the desired inequality. □

Letting $m \to \infty$ in (7.12), it follows from (2.38) that

$$\limsup_{n\to\infty} \frac{1}{n} \log \sup_E \sum_{x\in E} \exp \varphi_n(x) \le h_\mu(f,\xi) + \lim_{m\to\infty} \frac{1}{m} \int_X \varphi_m \, d\mu$$
$$\le h_\mu(f) + \lim_{m\to\infty} \frac{1}{m} \int_X \varphi_m \, d\mu,$$

where the supremum is taken over all (n,ε)-separated sets $E \subset X$. Therefore, letting $\varepsilon \to 0$ yields

$$P(\Phi) \le \sup_\nu \left(h_\nu(f) + \lim_{m\to\infty} \frac{1}{m} \int_X \varphi_m \, d\nu \right),$$

where the supremum is taken over all f-invariant probability measures ν in X. □

More generally, when the topological entropy is infinite the following statement was established in [42, 66]. Let

$$F(\mu) = \lim_{n\to\infty} \frac{1}{n} \int_X \varphi_n \, d\mu. \tag{7.14}$$

Theorem 7.2.4. *Let $f\colon X \to X$ be a continuous transformation of a compact metric space. For an asymptotically subadditive sequence Φ:*

1. *$P(\Phi) = -\infty$ if and only if $F(\mu) = -\infty$ for every $\mu \in \mathcal{M}_f$;*
2. *if $F(\mu) = -\infty$ for at least one measure $\mu \in \mathcal{M}_f$, then*

$$P(\Phi) = \sup \big\{ h_\mu(f) + F(\mu) : F(\mu) \ne -\infty \big\}.$$

7.3 Pressure for subadditive sequences

We consider in this section the particular case of a subadditive sequence Φ, and we describe two characterizations of the topological pressure $P(\Phi)$ obtained by Ban, Cao and Hu in [4] when the entropy is upper semicontinuous.

The first characterization expresses the topological pressure in terms of the classical pressure of the functions φ_n/n. The proof takes advantage of the variational principle in Theorem 7.2.1.

Theorem 7.3.1 ([4]). *Let $f\colon X \to X$ be a continuous transformation of a compact metric space with $h(f) < \infty$ such that the entropy map $\mu \mapsto h_\mu(f)$ is upper semicontinuous. For any subadditive sequence Φ, we have*

$$P(\Phi) = \lim_{n\to\infty} P(\varphi_n/n), \tag{7.15}$$

where $P(\varphi_n/n)$ is the classical topological pressure of the function φ_n/n.

Proof. The existence of the limit in (7.15) is due to Zhang [202]. We first observe that since the classical topological pressure $\varphi \mapsto P(\varphi)$ is convex (see for example [195]), and Φ is subadditive, we obtain

$$\begin{aligned} P\left(\frac{\varphi_{n+m}}{n+m}\right) &\le P\left(\frac{\varphi_n}{n+m} + \frac{\varphi_m \circ f^n}{n+m}\right) \\ &= P\left(\frac{n}{n+m} \cdot \frac{\varphi_n}{n} + \frac{m}{n+m} \cdot \frac{\varphi_m \circ f^n}{m}\right) \\ &\le \frac{n}{n+m} P\left(\frac{\varphi_n}{n}\right) + \frac{m}{n+m} P\left(\frac{\varphi_m \circ f^n}{m}\right) \\ &= \frac{n}{n+m} P\left(\frac{\varphi_n}{n}\right) + \frac{m}{n+m} P\left(\frac{\varphi_m}{m}\right). \end{aligned}$$

This shows that the sequence $nP(\varphi_n/n)$ is subadditive, and hence,

$$\lim_{n\to\infty} P\left(\frac{\varphi_n}{n}\right) = \inf_{n\in\mathbb{N}} P\left(\frac{\varphi_n}{n}\right). \tag{7.16}$$

Now we establish identity (7.15). We first show that

$$P(\Phi) \le \lim_{n\to\infty} P\left(\frac{\varphi_n}{n}\right).$$

For a fixed $m \in \mathbb{N}$, let us write $n = ms + l$, with $0 \le l \le m$. Since Φ is subadditive we obtain

$$\varphi_n(x) \le \frac{1}{m} \sum_{j=0}^{m-1} \sum_{i=0}^{s-2} \varphi_m(f^{im+j}(x)) + \frac{1}{m} \sum_{j=0}^{m-1} \left[\varphi_j(x) + \varphi_{m-j+l}(f^{(s-1)m+j}(x))\right].$$

Now we set $C = \max_{i=1,\ldots,2m-1} \|\varphi_i\|_\infty$. Then

$$\begin{aligned} \varphi_n(x) &\le \sum_{j=0}^{sm+l-1} \frac{1}{m} \varphi_m(f^j(x)) - \frac{1}{m} \sum_{j=(s-1)m}^{sm-1} \varphi_m(f^j(x)) + 2C \\ &\le \sum_{j=0}^{n-1} \frac{1}{m} \varphi_m(f^j(x)) + 4C, \end{aligned}$$

and hence, $P(\Phi) \le P(\varphi_m/m)$. Since m is arbitrary, we conclude that

$$P(\Phi) \le \lim_{m\to\infty} P\left(\frac{\varphi_m}{m}\right).$$

Now we establish the reverse inequality. Since the entropy is upper semicontinuous, for each $k \in \mathbb{N}$ there exists an f-invariant probability measure μ_k in X such that

$$P\left(\frac{\varphi_k}{k}\right) = h_{\mu_k}(f) + \int_X \frac{\varphi_k}{k}\, d\mu_k.$$

On the other hand, since Φ is subadditive, for each $m \in \mathbb{N}$ we have

$$\varphi_{km} \leq \sum_{j=0}^{k-1} \varphi_m \circ f^{kj},$$

and it follows from the f-invariance of the measures μ_{km} that

$$h_{\mu_{km}}(f) + \int_X \frac{\varphi_{km}}{km}\, d\mu_{km} \leq h_{\mu_{km}}(f) + \int_X \frac{\varphi_m}{m}\, d\mu_{km}.$$

Therefore,

$$\begin{aligned}
\lim_{n\to\infty} P\left(\frac{\varphi_n}{n}\right) &= \lim_{k\to\infty} P\left(\frac{\varphi_{km}}{km}\right) \\
&= \lim_{k\to\infty} \left(h_{\mu_{km}}(f) + \int_X \frac{\varphi_{km}}{km}\, d\mu_{km}\right) \\
&\leq \liminf_{k\to\infty} \left(h_{\mu_{km}}(f) + \int_X \frac{\varphi_m}{m}\, d\mu_{km}\right) \\
&\leq h_\mu(f) + \int_X \frac{\varphi_m}{m}\, d\mu,
\end{aligned} \tag{7.17}$$

where μ is any sublimit of the sequence $(\mu_{km})_{k\in\mathbb{N}}$. Now we observe that since the sequence $\int_X (\varphi_m/m)\, d\mu$ is subadditive, letting $m \to \infty$ in (7.17) yields

$$\lim_{n\to\infty} P\left(\frac{\varphi_n}{n}\right) \leq h_\mu(f) + \lim_{m\to\infty} \int_X \frac{\varphi_m}{m}\, d\mu \leq P(\Phi),$$

by the variational principle for the topological pressure in Theorem 7.2.1. This completes the proof of the theorem. □

The second characterization involves the classical pressures of the iterates f^n of the dynamics.

Theorem 7.3.2 ([4]). *Let $f\colon X \to X$ be a continuous transformation of a compact metric space with $h(f) < \infty$ such that the entropy map $\mu \mapsto h_\mu(f)$ is upper semicontinuous. For any subadditive sequence Φ, we have*

$$P(\Phi) = \lim_{k\to\infty} P_{f^k}(\varphi_k),$$

where $P_{f^k}(\varphi_k)$ is the classical topological pressure of the function φ_k computed with respect to f^k.

Proof. Given $m < k$, let us write $k = mq + r$, with $0 \leq r < m$. Setting

$$C = \max_{i=1,\dots,2m} \|\varphi_i\|_\infty,$$

it follows from the subadditivity of Φ that

$$\begin{aligned}\varphi_k(x) &\leq \frac{1}{m}\sum_{j=0}^{m-1}\sum_{i=0}^{q-2}\varphi_m\big(f^{im+j}(x)\big) + \frac{1}{m}\sum_{j=0}^{m-1}\big[\varphi_j(x) + \varphi_{m-j+l}(f^{(q-1)m+j}(x))\big] \\ &\leq \sum_{i=0}^{k-1}\frac{1}{m}\varphi_m(f^i(x)) + 4C.\end{aligned}$$

Hence, for each $n \in \mathbb{N}$ and $j = 0, \ldots, n-1$, we have

$$\varphi_k(f^{k_j}(x)) \leq \sum_{i=0}^{k-1}\frac{1}{m}\varphi_m(f^i(f^{k_j}(x))) + 4C,$$

and thus,

$$\sum_{j=0}^{n-1}\varphi_k(f^{k_j}(x)) \leq \sum_{i=0}^{nk-1}\frac{1}{m}\varphi_m(f^i(x)) + 4C. \tag{7.18}$$

Since any (n, ε)-separated set with respect to f^k is also an (nk, ε)-separated set with respect to f, it follows from (7.18) that

$$P_{f^k}(\varphi_k) \leq kP\left(\frac{\varphi_m}{m}\right) + 4C,$$

and hence,

$$\limsup_{k\to\infty}\frac{1}{k}P_{f^k}(\varphi_k) \leq P\left(\frac{\varphi_m}{m}\right)$$

for every $m \in \mathbb{N}$. By Theorem 7.3.1, we thus obtain

$$\limsup_{k\to\infty}\frac{1}{k}P_{f^k}(\varphi_k) \leq \lim_{m\to\infty} P\left(\frac{\varphi_m}{m}\right) = P(\Phi). \tag{7.19}$$

We recall that the existence of the limit in (7.19) was established in (7.16).

For the reverse inequality, given $k \in \mathbb{N}$ and $n > m$, let us write $n = km + r$, with $0 \leq r < k$. We also set

$$C = \max_{i=1,\ldots,k}\|\varphi_i\|_\infty.$$

By the uniform continuity of f, for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$, satisfying $\delta \to 0$ when $\varepsilon \to 0$, such that if E is an (n, ε)-separated set with respect to f, then it is also an (m, δ)-separated set with respect to f^k. On the other hand, since Φ is subadditive we have

$$\varphi_n(x) \leq \sum_{i=0}^{m-1}\varphi_j(f^{ik}(x)) + \varphi_r(f^{mk}(x)) \leq \sum_{i=0}^{m-1}(f^{ik}(x)) + C,$$

and hence,

$$P(\Phi) \le \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \sup_E \sum_{x\in E} \exp \sum_{i=0}^{m-1} \varphi_k(f^{ik}(x)),$$

where the supremum is taken over all (n,ε)-separated sets E with respect to f. By the initial observation, each of these sets is also an (m,δ)-separated set with respect to f^k, and hence,

$$\begin{aligned} P(\Phi) &\le \lim_{\delta\to 0} \limsup_{m\to\infty} \frac{1}{km+r} \log \sup_F \sum_{x\in F} \exp \sum_{i=0}^{m-1} \varphi_k(f^{ik}(x)) \\ &= \lim_{\delta\to 0} \limsup_{m\to\infty} \frac{1}{km} \log \sup_F \sum_{x\in F} \exp \sum_{i=0}^{m-1} \varphi_k(f^{ik}(x)) \\ &= \frac{1}{k} P_{f^k}(\varphi_k), \end{aligned}$$

where the supremum is taken over all (m,δ)-separated sets F with respect to f^k. Letting $k\to\infty$ yields

$$P(\Phi) \le \liminf_{k\to\infty} \frac{1}{k} P_{f^k}(\varphi_k).$$

Together with (7.19) this establishes the desired result. □

7.4 Equilibrium measures

We discuss in this section the existence of equilibrium measures for an arbitrary asymptotically subadditive sequence. In a similar manner to that in the classical theory (see Definition 2.4.1), equilibrium measures are those at which the supremum in (7.8) is attained.

More precisely, let $f\colon X\to X$ be a continuous transformation of a compact metric space.

Definition 7.4.1. Given an asymptotically subadditive sequence Φ of continuous functions $\varphi_n\colon X\to X$, an f-invariant probability measure μ in X is called an *equilibrium measure* for Φ (with respect to f) if

$$P(\Phi) = h_\mu(f) + \lim_{n\to\infty} \frac{1}{n} \int_X \varphi_n \, d\mu.$$

We want to show that asymptotically subadditive sequences have equilibrium measures. For the formulation of the result we recall that given a closed convex set $C\subset\mathbb{R}^k$, a point $x\in C$ is called an *extreme point* of C if it cannot be written as a proper convex combination of two distinct points in C, that is, we have $x=y=z$ whenever $x=ty+(1-t)z$ for some $y,z\in C$ and $t\in(0,1)$.

The existence of equilibrium measures for continuous transformations with upper semicontinuous entropy was established by Feng and Huang.

Theorem 7.4.2 ([66]). *Let $f\colon X \to X$ be a continuous transformation of a compact metric space with $h(f) < \infty$ such that the entropy map $\mu \mapsto h_\mu(f)$ is upper semicontinuous. If Φ is an asymptotically subadditive sequence with*

$$\limsup_{n\to\infty} \frac{1}{n} \sup_{x\in X} \varphi_n(x) \neq -\infty, \tag{7.20}$$

then

1. *the set $E_\Phi \subset \mathcal{M}_f$ of all equilibrium measures for Φ is a nonempty compact convex set;*
2. *each extreme point of E_Φ is an ergodic measure;*
3. *for each $q > 0$ we have*
$$[p'(q^-), p'(q^+)] = \{F(\mu) : \mu \in E_{q\Phi}\},$$
with F as in (7.14) and with the function $p\colon \mathbb{R} \to \mathbb{R}$ defined by $p(q) = P(q\Phi)$.

Proof. Since $h(f) < \infty$, one can use the variational principle in Theorem 7.2.1 together with condition (7.20) to show that the function p takes only finite values. Moreover, p is continuous and convex in $\mathbb{R}^+$. For the arguments, we refer to the proof of Theorem 9.3.2 which considers the more general case of maps for which the entropy is not necessarily upper semicontinuous.

In order to show that $E_\Phi \neq \varnothing$ we need an auxiliary result.

Lemma 7.4.3. *The map $F\colon \mathcal{M}_f \to \mathbb{R} \cup \{-\infty\}$ is upper semicontinuous.*

Proof of the lemma. Let $(\mu_n)_{n\in\mathbb{N}} \subset \mathcal{M}_f$ be a sequence of measures converging to a measure μ. Given $\varepsilon > 0$, there exists a subadditive sequence $(\psi_n)_{n\in\mathbb{N}}$ and $n_0 \in \mathbb{N}$ such that $\|\varphi_n - \psi_n\|_\infty < n\varepsilon$ for $n \geq n_0$. For each $n \geq n_0$, we have

$$\begin{aligned}
\limsup_{m\to\infty} F(\mu_m) &\leq \limsup_{m\to\infty} \lim_{n\to\infty} \frac{1}{n} \int_X \psi_n \, d\mu_m + \varepsilon \\
&= \limsup_{m\to\infty} \inf_{n\in\mathbb{N}} \frac{1}{n} \int_X \psi_n \, d\mu_m + \varepsilon \\
&\leq \limsup_{m\to\infty} \frac{1}{n} \int_X \psi_n \, d\mu_m + \varepsilon \\
&= \frac{1}{n} \int_X \psi_n \, d\mu + \varepsilon.
\end{aligned}$$

Letting $n \to \infty$, we obtain

$$\limsup_{m\to\infty} F(\mu_m) \leq \lim_{n\to\infty} \frac{1}{n} \int_X \psi_n \, d\mu + \varepsilon \leq F(\mu) + 2\varepsilon,$$

and it follows from the arbitrariness of ε that

$$\limsup_{m\to\infty} F(\mu_m) \leq F(\mu).$$

This completes the proof of the lemma. □

By Theorem 7.2.1, there exists a sequence $(\mu_n)_{n\in\mathbb{N}} \subset \mathcal{M}_f$ such that

$$P(\Phi) = \lim_{n\to\infty} \big(h_{\mu_n}(f) + F(\mu_n)\big).$$

Now let μ be any sublimit of this sequence. It follows from the upper semicontinuity of the maps $\mu \mapsto h_\mu(f)$ (by hypothesis) and $\mu \mapsto F(\mu)$ (by Lemma 7.4.3) that

$$P(\Phi) \le h_\mu(f) + F(\mu).$$

By Theorem 7.2.1, we conclude that μ is an equilibrium measure for Φ, and thus $E_\Phi \neq \varnothing$. An analogous argument shows that any limit point of E_Φ is also in E_Φ. Therefore, the set E_Φ is compact. To show that E_Φ is convex, let us take $\mu_1, \mu_2 \in E_\Phi$ and $p \in [0,1]$. For the measure $\mu = p\mu_1 + (1-p)\mu_2$, we have

$$h_\mu(f) = ph_{\mu_1}(f) + (1-p)h_{\mu_2}(f) \tag{7.21}$$

and

$$F(\mu) = pF(\mu_1) + (1-p)F(\mu_2). \tag{7.22}$$

Therefore,

$$\begin{aligned} h_\mu(f) + F(\mu) &= p\big(h_{\mu_1}(f) + F(\mu_1)\big) + (1-p)\big(h_{\mu_2}(f) + F(\mu_2)\big) \\ &= pP(\Phi) + (1-p)P(\Phi) = P(\Phi). \end{aligned}$$

This shows that $\mu \in E_\Phi$, and hence, the set E_Φ is convex.

Now we assume that μ is an extreme point of E_Φ. If $\mu = p\mu_1 + (1-p)\mu_2$ for some $\mu_1, \mu_2 \in \mathcal{M}_f$ and $p \in (0,1)$, then by (7.21) and (7.22) we obtain

$$\begin{aligned} P(\Phi) &= h_\mu(f) + F(\mu) \\ &= p\big(h_{\mu_1}(f) + F(\mu_1)\big) + (1-p)\big(h_{\mu_2}(f) + F(\mu_2)\big). \end{aligned} \tag{7.23}$$

On the other hand, by Theorem 7.2.1, we have

$$h_{\mu_i}(f) + F(\mu_i) \ge P(\Phi)$$

for $i = 1, 2$, and it follows from (7.23) that

$$h_{\mu_i}(f) + F(\mu_i) = P(\Phi)$$

for $i = 1, 2$. That is, μ_1 and μ_2 are equilibrium measures. Since μ is an extreme point of E_Φ, we conclude that $\mu = \mu_1 = \mu_2$. Thus, μ is also an extreme point of $\mathcal{M}_f$. In other words, μ is an ergodic measure.

For the last property, let us consider the function $f\colon \mathbb{R}^+ \times \mathcal{M}_f \to \mathbb{R}$ defined by

$$f(q, \mu) = h_\mu(f) + qF(\mu). \tag{7.24}$$

We note that f is upper semicontinuous, and that

$$p(q) := \sup_{\mu\in\mathcal{M}_f} f(q, \mu) > -\infty$$

for every $q > 0$. Moreover, $p\colon \mathbb{R}^+ \to \mathbb{R}$ is a convex function, and

$$I(q) = \big\{\mu \in \mathcal{M}_f : f(q,\mu) = p(q)\big\} \tag{7.25}$$

is a nonempty compact convex set for each $q > 0$. For the convexity of p, we note that given $q, q' > 0$ and $t \in (0,1)$, it follows from Theorem 7.2.1 that

$$\begin{aligned} p(tq + (1-t)q') &= \sup_\mu \big(h_\mu(f) + (tq + (1-t)q')F(\mu)\big) \\ &= \sup_\mu \big(h_\mu(f) + tqF(\mu) + (1-t)q'F(\mu)\big) \\ &\le t\sup_\mu (h_\mu(f) + qF(\mu)) + (1-t)\sup_\mu (h_\mu(f) + q'F(\mu)) \\ &= tp(q) + (1-t)p(q'). \end{aligned}$$

Now we set

$$R(q) = \big\{F(\mu) : \mu \in E_{q\Phi}\big\} \subset \mathbb{R}.$$

One can easily verify that $R(q)$ is a nonempty convex set, and thus it is an interval.

Lemma 7.4.4. *For each $q > 0$:*

1. *the set $R(q)$ is compact;*
2. *for each $\delta > 0$ there exists $\gamma > 0$ such that $R(t) \subset B_\delta(R(q))$ for every $t > 0$ with $|t - q| < \gamma$, where*

$$B_\delta(R(q)) = \big\{s \in \mathbb{R} : \operatorname{dist}(s, R(q)) \le \delta\big\}.$$

Proof of the lemma. Given $(a_n)_{n\in\mathbb{N}} \subset R(q)$, for each $n \in \mathbb{N}$ let us take $\mu_n \in I(q)$ such that $a_n = F(\mu_n)$. For each $t > 0$, we have

$$f(t,\mu_n) - p(q) = f(t,\mu_n) - f(q,\mu_n) \ge a_n(t-q). \tag{7.26}$$

This implies that the sequence $(a_n)_{n\in\mathbb{N}}$ is bounded, since otherwise there would exist $t > 0$ such that $(a_n(t-q))_{n\in\mathbb{N}}$ is unbounded from above, but

$$f(t,\mu_n) - f(q,\mu_n) = f(t,\mu_n) - p(q) \le p(t) - p(q).$$

Therefore if necessary taking subsequences, we may assume that $\mu_n \to \mu$ and $a_n \to a$ when $n \to \infty$, for some $\mu \in I(q)$ and $a \in \mathbb{R}$. Since $f(t,\cdot)$ is upper semicontinuous, it follows from (7.26) that

$$f(t,\mu) - f(q,\mu) = f(t,\mu) - p(q) \ge a(t-q)$$

for each $t > 0$. Therefore, $a \in R(q)$, and hence, the set $R(q)$ is compact. To establish the second property we proceed by contradiction. Otherwise, there would exist $\delta > 0$ and a sequence $(t_n)_{n\in\mathbb{N}} \subset \mathbb{R}^+$ with $t_n \to q$ when $n \to \infty$, as well as $a_n \in R(t_n)$ satisfying

$$\operatorname{dist}(a_n, R(q)) > \delta \quad \text{for each} \quad n \in \mathbb{N}. \tag{7.27}$$

Now let us take $\mu_n \in I(t_n)$ so that $a_n = F(\mu_n)$. Then

$$f(t, \mu_n) - p(t_n) = f(t, \mu_n) - f(t_n, \mu_n) \geq a_n(t - t_n) \tag{7.28}$$

for each $t > 0$, and one can show in a similar manner that the sequence $(a_n)_{n\in\mathbb{N}}$ is bounded. Therefore, if necessary taking subsequences, we may assume that $\mu_n \to \mu$ and $a_n \to a$ when $n \to \infty$, for some $\mu \in \mathcal{M}_f$ and $a \in \mathbb{R}$. By the upper semicontinuity of f and the continuity of p, we have

$$f(q, \mu) \geq \limsup_{n\to\infty} f(t_n, \mu_n) = \limsup_{n\to\infty} p(t_n) = p(q).$$

Hence, $\mu \in I(q)$. Letting $n \to \infty$ in (7.28) thus yields

$$\begin{aligned} f(t, \mu) - f(q, \mu) &= f(t, \mu) - p(q) \\ &\geq \limsup_{n\to\infty} \big(f(t, \mu_n) - p(t_n)\big) \geq a(t - q) \end{aligned}$$

for each $t > 0$, and hence $a = F(\mu)$. Therefore, $a \in R(q)$, which contradicts property (7.27). This completes the proof of the lemma. □

We note that the second statement in Lemma 7.4.4 corresponds to a certain continuity of the intervals $R(q)$ with respect to q.

We proceed with the proof of the theorem. For each $a \in R(q)$, there exists $\mu \in I(q)$ such that $a = F(\mu)$. Hence, for each $t > 0$ we have

$$\begin{aligned} p(t) - p(q) &\geq f(t, \mu) - p(q) \\ &= f(t, \mu) - f(q, \mu) \geq a(t - q). \end{aligned}$$

This implies that $a \in A_q := [p'(q^-), p'(q^+)]$, and thus $R(q) \subset A_q$.

Now we establish the reverse inclusion, that is, $A_q \subset R(q)$. Otherwise, let us take $a \in A_q \setminus R(q)$. Since, $R(t) \subset A_t$, we have

$$p(t) - p(q) \leq b(t - q) \quad \text{for} \quad t > 0,\ b \in R(t).$$

Moreover, since $a \in A_q$, we also have

$$p(t) - p(q) \geq a(t - q) \quad \text{for} \quad t > 0.$$

Therefore,

$$a(t - q) \leq b(t - q) \quad \text{for} \quad t > 0,\ b \in R(t). \tag{7.29}$$

Since $a \notin R(q)$ and $R(q)$ is compact, there exists $\delta > 0$ such that $a \notin B_\delta(R(q))$. Now we observe that by Lemma 7.4.4, there exists $\gamma > 0$ such that $R(t) \subset B_\delta(R(q))$

whenever $|t-q| \le \gamma$. Let us take a nonzero $\gamma' \in (-\gamma, \gamma)$ such that $(a-b)\operatorname{sgn}\gamma' > 0$ for every $b \in B_\delta(R(q))$, and $t_0 := q + \gamma'/2 > 0$. Then

$$a(t_0 - q) > b(t_0 - q) \quad \text{for every} \quad b \in R(t_0),$$

which contradicts (7.29). This shows that $A_q \subset R(q)$, and the proof of the theorem is complete. □

7.5 The case of symbolic dynamics

We consider briefly in this section the particular case of symbolic dynamics, and we survey some of the existing results in the literature.

For a certain class of subadditive sequences in a topological Markov chain, the variational principle in (7.8) was first established by Falconer [56]. His result can be formulated as follows.

Theorem 7.5.1. *Let Φ be a subadditive sequence of functions $\varphi_n \colon \Sigma_A^+ \to \mathbb{R}$ satisfying the following properties:*

1. *there exists $K > 0$ such that $\gamma_n(\Phi, \mathcal{U}) \le K$ for every $n \in \mathbb{N}$, where $\mathcal{U}$ is the cover of Σ_A^+ formed by the 1-cylinder sets;*
2. *there exists $L > 0$ such that*

$$|\varphi_n(x)|/n \le L \quad \textit{and} \quad |\varphi_n(x) - \varphi(y)|/n \le L$$

for every $n \in \mathbb{N}$ and $x, y \in \Sigma_A^+$.

Then

$$P(\Phi) = \sup_\mu \left(h_\mu(f) + \lim_{n\to\infty} \frac{1}{n} \int_{\Sigma_A^+} \varphi_n \, d\mu \right),$$

where the supremum is taken over all σ-invariant probability measures μ in Σ_A^+.

In the remainder of the section we describe some results by Käenmäki [106] and by Feng and Käenmäki [67] concerning the construction of equilibrium measures for a particular class of subadditive sequences in the case of symbolic dynamics. These sequences are well adapted to the study of the dimension of a class of limit sets of iterated function systems (see [106]) and of the multifractal analysis of the top Lyapunov exponent of products of matrices (see [63, 65, 68]).

We first introduce some notation. Given $\kappa \in \mathbb{N}$, we write $\Sigma^n = \{1, \ldots, \kappa\}^n$ for each $n \in \mathbb{N}$ and $|\omega| = n$ for each $\omega \in \Sigma^n$. We also write $\Sigma^* = \bigcup_{n\in\mathbb{N}} \Sigma^n$. We denote by $\mathcal{C}$ the class of all (parameterized) functions $\psi_\omega^t \colon \Sigma_\kappa^+ \to \mathbb{R}^+$, for $t \ge 0$ and $\omega \in \Sigma^*$, with $\psi_\omega^0 = 1$ and satisfying the following properties:

1. there exists $K_t > 0$ such that $\psi_\omega^t(\omega_1) \le K \psi_\omega^t(\omega_2)$ for every $\omega_1, \omega_2 \in \Sigma_\kappa^+$;

2. for every $\omega' \in \Sigma_\kappa^+$ and $j \in [1, |\omega|] \cap \mathbb{N}$ we have
$$\psi_\omega^t(\omega') \le \psi_{\omega|j}^t(\sigma^j(\omega)\omega')\psi_{\sigma^j(\omega)}^t(\omega'), \tag{7.30}$$
where $\omega|j$ are the first j elements of ω, and where $\sigma^j(\omega)\omega'$ denotes the juxtaposition of the sequences $\sigma^j(\omega)$ and ω';

3. for each $\delta > 0$ there exist $a = a(\delta), b = b(\delta) \in (0,1)$, with $a(\delta) \nearrow 1$ and $b(\delta) \nearrow 1$ when $\delta \to 0$, such that
$$\psi_\omega^t(\omega')a^{|\omega|} \le \psi_\omega^{t+\delta}(\omega') \le \psi_\omega^t(\omega')b^{|\omega|}$$
for every $\omega' \in \Sigma_\kappa^+$.

We note that this class of (parameterized) functions contains some particular classes earlier considered by Falconer [55, 59] in connection with the study of the dimension of repellers of nonconformal transformations.

For each function in the class $\mathcal{C}$, using the subadditivity in (7.30) it is shown in [106] that given $\omega' \in \Sigma_\kappa^+$ and a σ-invariant probability measure μ in Σ_κ^+, the limits
$$p(t) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega\in\Sigma^n} \psi_\omega^t(\omega') \tag{7.31}$$
and
$$s_\mu(t) = \lim_{n\to\infty} \frac{1}{n} \sum_{\omega\in\Sigma^n} \mu(C_\omega) \log \psi_\omega^t(\omega')$$
exist, where $C_\omega \subset \Sigma_\kappa^+$ is the set of sequences whose first n elements are equal to the first n elements of ω. Moreover, the two numbers are independent of ω'.

Now we verify that the limit $p(t)$ is a particular case of the nonadditive topological pressure. For this, given $\omega' \in \Sigma_\kappa^+$ and $n \in \mathbb{N}$, we define a sequence of functions $\varphi_n^t \colon \Sigma_\kappa^+ \to \mathbb{R}$ by
$$\varphi_n^t(\omega) = \sup_{\omega''\in C_\omega} \log \psi_{\omega''}^t(\omega'). \tag{7.32}$$
The first condition on the class $\mathcal{C}$ ensures that the sequence $\Phi^t = (\varphi_n^t)_{n\in\mathbb{N}}$ has tempered variation (see (4.2)). Moreover, by the second condition, it follows from Theorems 4.2.6 and 4.5.1 that $p(t)$ coincides with the nonadditive topological pressure $P_{\Sigma_\kappa^+}(\Phi^t)$ of the sequence Φ^t for any ω'. Finally, by the third condition on $\mathcal{C}$ we can readily apply Theorem 4.4.2 to conclude that there is a unique $t \ge 0$ such that $p(t) = 0$. We also note that Kingman's subadditive ergodic theorem can be used to show that
$$s_\mu(t) = \lim_{n\to\infty} \frac{1}{n} \int_{\Sigma_\kappa^+} \varphi_n^t \, d\mu.$$

Now we observe that by the first condition on the class $\mathcal{C}$, the first condition in Theorem 7.5.1 holds. Thus, by this theorem (or by Theorem 7.2.1) the variational principle holds, that is,

$$P_{\Sigma_\kappa^+}(\Phi^t) = \sup_\mu \left(h_\mu(\sigma) + \lim_{n\to\infty} \frac{1}{n} \int_{\Sigma_\kappa^+} \varphi_n^t \, d\mu \right),$$

where the supremum is taken over all σ-invariant probability measures μ in Σ_κ^+. Using the above notation, this identity can be written in the form

$$p(t) = \sup_\mu \left(h_\mu(\sigma) + s_\mu(t) \right). \tag{7.33}$$

In the present context the existence of equilibrium measures, that is, measures at which the supremum in (7.33) is attained, was first establish by Käenmäki [106] (the result is a special case of Theorem 7.4.2).

Now we consider a particular class of functions in $\mathcal{C}$ that are obtained from products of matrices. Given $p, m \in \mathbb{N}$, let $M_1, \ldots, M_p$ be $m \times m$ matrices. For each $t > 0$, $n \in \mathbb{N}$, and $\omega \in \Sigma^n$, we consider the constant function

$$\bar\psi_\omega^t = \|M_{i_1} \cdots M_{i_n}\|^t,$$

where $\omega = (i_1 \cdots i_n)$. Again we define a sequence $\bar\Phi^t$ as in (7.32), that is,

$$\bar\varphi_n^t(\omega) = \sup_{\omega'' \in C_\omega} \log \bar\psi_{\omega''}^t(\omega') = \sup_{\omega'' \in C_\omega} \log \|M_{i_1} \cdots M_{i_n}\|^t,$$

where $\omega'' = (i_1 \cdots i_n)$. One can easily verify that the functions $\bar\psi_\omega^t$ belong to the class $\mathcal{C}$, and that the number $p(t)$ in (7.31) is given by

$$p(t) = \lim_{n\to\infty} \frac{1}{n} \sum_{\omega \in \Sigma^n} \|M_{i_1} \cdots M_{i_n}\|^t.$$

Moreover, given a σ-invariant probability measure μ in Σ_κ^+, we have

$$s_\mu(t) = t \lim_{n\to\infty} \frac{1}{n} \sum_{\omega \in \Sigma^n} \mu(C_\omega) \log \|M_{i_1} \cdots M_{i_n}\|.$$

The following result is due to Feng and Käenmäki.

Theorem 7.5.2 ([67]). *If for each $n \in \mathbb{N}$ there exist $i_1, \ldots, i_n \in \{1, \ldots, m\}$ such that $M_{i_1} \cdots M_{i_n} \neq 0$, then for each $t \geq 0$ there are at most m ergodic equilibrium measures for the sequence $\bar\Phi^t$. If in addition the only proper vector space V such that $M_i V \subset V$ for $i = 1, \ldots, m$ is the origin, then for each $t \geq 0$ there is a unique equilibrium measure for the sequence $\bar\Phi^t$.*

The condition in Theorem 7.5.2 concerning the subspaces V is used in [65] to show that there exist $c > 0$ and $k \in \mathbb{N}$ such that for each $\omega, \omega' \in \Sigma^*$ there exists $\bar\omega \in \bigcup_{j=1}^k \Sigma^j$ for which

$$\|M_{\omega\bar\omega\omega'}\| \geq c \|M_\omega\| \cdot \|M_{\omega'}\|. \tag{7.34}$$

It is essentially this property that allows one to establish the existence of a unique equilibrium measure in [67]. We note that property (7.34) ensures that the sequence $\bar{\Phi}^t$ is almost additive (see Definition 10.1.1), and thus the existence of a unique ergodic measure in Theorem 7.5.2, as well as its Gibbs property (also obtained in [67]), follow from earlier results in [6] for the class of almost additive sequences (see Theorem 10.1.9).

Chapter 8

Limit Sets of Geometric Constructions

We consider in this chapter limits sets of geometric constructions, mostly from the point of view of the dimension theory of dynamical systems. A simple example is the middle-third Cantor set, which is a repeller of a piecewise-linear expanding map of the interval. Roughly speaking, a geometric construction corresponds to the geometric structure provided by the intervals in the construction of the middle-third Cantor set or more generally by the rectangles of any Markov partition of a repeller, although not necessarily determined by some dynamics. More precisely, geometric constructions are defined in terms of certain decreasing sequences of compact sets, such as the intervals of decreasing size in the construction of the middle-third Cantor set. Moreover, even when one can define naturally an induced map for which the limit set of the geometric construction is an invariant set, this map need not be expanding. Our main aim is to describe how the theory for repellers developed in Chapter 5 can be extended to this more general setting, with emphasis on the case when the associated sequences are subadditive.

8.1 Geometric constructions

We consider in this section a class of geometric constructions defined by an arbitrary symbolic dynamics. More precisely, we consider the class of the so-called generalized Moran constructions, which can be thought of as geometric constructions defined by balls. This is a weak counterpart of the conformality property in the case of repellers. On the other hand, the radii of the balls may be arbitrary.

We first recall the notion of geometric construction.

Definition 8.1.1. A *geometric construction* in $\mathbb{R}^m$ (see Figure 8.1) is defined by:

1. a compact shift-invariant set $Q \subset \Sigma_\kappa^+$ for some positive integer κ;

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2. a decreasing sequence of closed sets $\Delta_{i_1 \cdots i_n} \subset \mathbb{R}^m$ for each $(i_1 i_2 \cdots) \in Q$ and $n \in \mathbb{N}$, with $\operatorname{diam} \Delta_{i_1 \cdots i_n} \to 0$ when $n \to \infty$.

We also say that the geometric construction is modeled by Q.

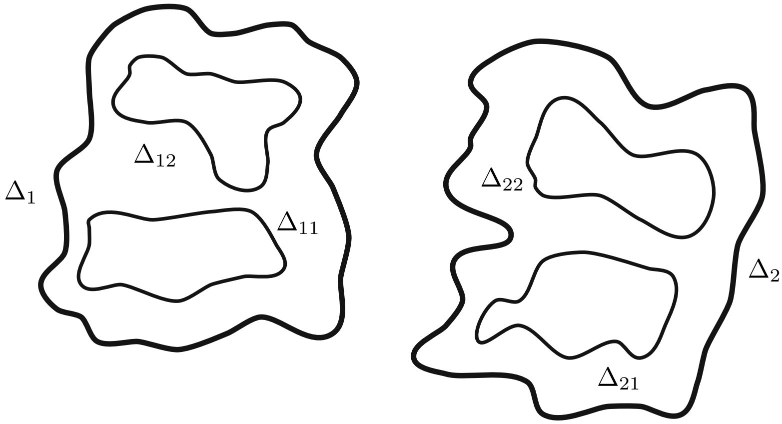

Figure 8.1: Geometric construction in $\mathbb{R}^2$ modeled by Σ_2^+

The sets $\Delta_{i_1 \cdots i_n}$ are usually called *basic sets*. We notice that they may not have regular boundaries, and that they may even be Cantor sets. Moreover, they need not be disjoint at each step of the construction.

The *limit set* of a geometric construction is the compact set defined by

$$F = \bigcap_{n=1}^{\infty} \bigcup_{\gamma \in Q_n} \Delta_\gamma,$$

where Q_n is the set of all Q-admissible sequences of length n, that is, the sequences $(i_1 \cdots i_n)$ such that

$$(j_1 \cdots j_n) = (i_1 \cdots i_n) \quad \text{for some sequence} \quad (j_1 j_2 \cdots) \in Q.$$

For each $\omega = (i_1 i_2 \cdots) \in Q$, we define the set

$$\chi(\omega) = \bigcap_{n=1}^{\infty} \Delta_{i_1 \cdots i_n}. \tag{8.1}$$

By condition 2 in the definition of geometric construction, the set $\chi(\omega)$ consists of a single point in F, and thus (8.1) defines the *coding map* $\chi \colon Q \to F$ for the limit set. We note that χ is onto, by the construction of F, although it may not be injective. Moreover, one can easily verify that the map χ is continuous.

Now we consider a particular class of geometric constructions.

Definition 8.1.2. A geometric construction is said to be a *generalized Moran construction* if there exist balls $\underline{B}_{i_1\cdots i_n}$ and $\overline{B}_{i_1\cdots i_n}$ for each $(i_1 i_2 \cdots) \in Q$ and $n \in \mathbb{N}$, and positive constants $C_1 \leq C_2$ such that:

1. $\underline{B}_{i_1\cdots i_n} \subset \Delta_{i_1\cdots i_n} \subset \overline{B}_{i_1\cdots i_n}$;
2. the interiors of the sets $\underline{B}_{i_1\cdots i_n}$ and $\underline{B}_{j_1\cdots j_m}$ are disjoint for any $(i_1 \cdots i_n) \neq (j_1 \cdots j_n)$ and $m \geq n$;
3. the radii of $\underline{B}_{i_1\cdots i_n}$ and $\overline{B}_{i_1\cdots i_n}$ are respectively $C_1 r_{i_1\cdots i_n}$ and $C_2 r_{i_1\cdots i_n}$ for some positive number $r_{i_1\cdots i_n}$.

We notice that the numbers $r_{i_1\cdots i_n}$ may satisfy no asymptotic behavior. See Figure 8.2 for an example of a generalized Moran construction where each basic set $\Delta_{i_1\cdots i_n}$ is a ball.

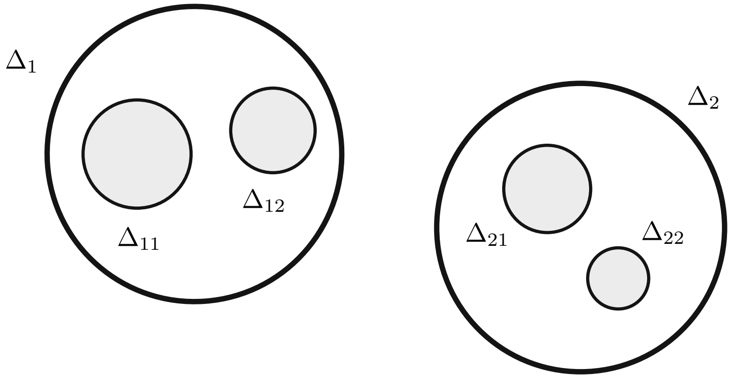

Figure 8.2: Generalized Moran construction

Given a generalized Moran construction, we consider the sequence of functions $\varphi_n \colon Q \to \mathbb{R}$ defined by

$$\varphi_n(i_1 i_2 \cdots) = \log r_{i_1\cdots i_n},$$

and we denote it by Φ. We note that Φ has tempered variation (see (4.2)). By Theorem 4.1.2, for each $s \in \mathbb{R}$ one can compute the nonadditive topological pressure $P_Q(s\Phi)$, and the nonadditive lower and upper capacity topological pressures $\underline{P}_Q(s\Phi)$ and $\overline{P}_Q(s\Phi)$, of the sequence of functions $s\Phi$ in the set Q (with respect to the shift map).

Now we assume that there exist constants $\lambda_1, \lambda_2 \in (0,1)$ such that

$$\lambda_1^n \leq r_{i_1\cdots i_n} \leq \lambda_2^n \tag{8.2}$$

for every $(i_1 i_2 \cdots) \in Q$ and $n \in \mathbb{N}$. By (8.2) and the second property in Theorem 4.4.1, there exist unique roots s_P, $s_{\underline{P}}$, and $s_{\overline{P}}$ respectively of the equations

$$P_Q(s\Phi) = 0, \quad \underline{P}_Q(s\Phi) = 0, \quad \text{and} \quad \overline{P}_Q(s\Phi) = 0.$$

Theorem 8.1.3 ([5]). *For a generalized Moran construction satisfying* (8.2)*:*

1. *we have*
$$s_P \le \dim_H F \le \underline{\dim}_B F \le \overline{\dim}_B F \le s_{\overline{P}};$$

2. *if the sequence* Φ *is subadditive, then*
$$\dim_H F = \underline{\dim}_B F = \overline{\dim}_B F = s_P = s_{\underline{P}} = s_{\overline{P}} = s,$$
where s *is the unique root of the equation*
$$\lim_{n\to\infty} \frac{1}{n} \log \sum_{(i_1\cdots i_n)\in Q_n} r^s_{i_1\cdots i_n} = 0.$$

Proof. Let $\mathcal{U}_n$ be the open cover of the space $X = \Sigma_\kappa^+$ formed by the n-cylinder sets. We note that $\operatorname{diam}\mathcal{U}_n \to 0$ when $n \to \infty$. By Theorem 4.1.2, for each $s \in \mathbb{R}$ we have
$$P_Q(s\Phi) = \lim_{n\to\infty} P_Q(s\Phi, \mathcal{U}_n).$$

Given $\varepsilon > 0$, we have $m_H(F, \dim_H F + \varepsilon) = 0$. Hence, for each $\delta > 0$, there exist a cover $\mathcal{U}$ of F composed of balls, such that
$$\sum_{U\in\mathcal{U}} (\operatorname{diam}\mathcal{U})^{\dim_H F+\varepsilon} < \delta.$$

For each $U \in \mathcal{U}$, we consider the cover $\Gamma_U \subset \bigcup_{n\in\mathbb{N}} \mathcal{W}_n(\mathcal{U}_1)$ of the set $F \cap U$ composed of the vectors $U \in \bigcup_{n\in\mathbb{N}} \mathcal{W}_n(\mathcal{U}_1)$ such that $X(U) = C_{i_1\cdots i_n}$, where $C_{i_1\cdots i_n}$ is some cylinder set for which
$$r_{i_1\cdots i_n} \le \operatorname{diam} U \le r_{i_1\cdots i_{n-1}} \quad \text{and} \quad \Delta_{i_1\cdots i_n} \cap U \ne \varnothing.$$

Then there exists $C > 0$ such that $\operatorname{card}\Gamma_U \le C$ for every $U \in \mathcal{U}$.

For each $U \in \mathcal{U}$ and $l \in \mathbb{N}$, let $U_l \in \bigcup_{n\in\mathbb{N}} \mathcal{W}_n(\mathcal{U}_l)$ be the unique vector of length $m(U) - l$ such that $X(U_l) = X(U)$. Then
$$\Gamma_l = \big\{U_l : U \in \Gamma_U \text{ for some } \ U \in \mathcal{U}\big\}$$

is a cover of F, and we obtain
$$\begin{aligned}\sum_{U_l\in\Gamma_l} (r_{X(U_l)})^{\dim_H F+\varepsilon} &\le \sum_{U\in\mathcal{U}} \sum_{U_l\in\Gamma_l : U\in\Gamma_U} (r_{X(U)})^{\dim_H F+\varepsilon} \\ &\le \sum_{U\in\mathcal{U}} C(r_{X(U)})^{\dim_H F+\varepsilon} < C\delta.\end{aligned}$$

Hence,
$$M_Q(0, (\dim_H F + \varepsilon)\Phi, \mathcal{U}_l) < C\delta,$$

and $M_Q(0, (\dim_H F + \varepsilon)\Phi, \mathfrak{U}_l) = 0$, because δ is arbitrary. We conclude that

$$P_Q((\dim_H F + \varepsilon)\Phi, \mathfrak{U}_l) \leq 0$$

for each $l \in \mathbb{N}$. Therefore, $P_Q((\dim_H F + \varepsilon)\Phi) \leq 0$, and $\dim_H F + \varepsilon \geq s_P$. Since ε is arbitrary, we obtain $\dim_H F \geq s_P$.

Now we establish an upper bound for the upper box dimension. For this, we introduce a special open cover of F for each sufficiently small $r > 0$. Given $\omega = (i_1 i_2 \cdots) \in Q$, let $n(\omega)$ be the unique positive integer such that

$$r_{i_1 \cdots i_{n(\omega)}} < r \leq r_{i_1 \cdots i_{n(\omega)-1}}$$

(condition 2 in the definition of geometric construction ensures that $n(\omega)$ is well defined for each sufficiently small r). We denote by $\Delta(\omega)$ the largest basic set in the geometric construction such that:

1. $\chi(\omega) \in \Delta(\omega)$;
2. $\Delta_{i_1 \cdots i_{n(\omega')}} \subset \Delta(\omega)$ for each $\chi(\omega') \in \Delta(\omega)$;
3. there exists $\chi(\omega') \in \Delta(\omega)$ for which $\Delta(\omega) = \Delta_{i_1 \cdots i_{n(\omega')}}$.

The sets $\Delta(\omega)$ corresponding to different $\omega \in Q$ shall be denoted by Δ_r^j, for $j = 1, \ldots, N_r$, and they form a cover of F. By the third property, there exists $\omega_j \in Q$ such that

$$\Delta_r^j = \Delta_{i_1(\omega_j) \cdots i_{n(\omega_j)}(\omega_j)},$$

for each j. Moreover, $n(\omega_j) \leq \log r / \log \lambda_2 + 1$ for $j = 1, \ldots, N_r$. Since

$$\operatorname{diam}(C \cap \Delta_r^j) \leq C_2 r_{i_1(\omega_j) \cdots i_{n(\omega_j)}(\omega_j)} < C_2 r,$$

we obtain

$$\sum_{m \in \mathbb{N}} \operatorname{card}\big\{j : n(\omega_j) = m\big\} \geq N_{C_2 r}(F).$$

Hence, there exists $m(r) \in \mathbb{N}$ such that

$$\operatorname{card}\big\{j : n(\omega_j) = m(r)\big\} \geq \frac{N_{C_2 r}(F)}{\log r / \log \lambda_2 + 1} \geq K' \frac{N_{C_2 r}(F)}{-\log r},$$

where $K' = -\log \lambda_2 / 2$. On the other hand, given $\delta > 0$ there exists a sequence $r_n \searrow 0$ when $n \to \infty$ such that

$$N_{C_2 r_n}(F) > r_n^{\delta - \overline{\dim}_B F} \quad \text{for} \quad n \in \mathbb{N}.$$

We notice that $1/(-\log r) \geq r^\delta$ for all sufficiently small $r > 0$. Now we set $m_n = m(r_n) - 1$. For each $s \leq \overline{\dim}_B F - 2\delta$, we obtain

$$\begin{aligned}
\kappa \sum_{(i_1 \cdots i_{m_n}) \in Q_{m_n}} r_{i_1 \cdots i_{m_n}}^s &\geq \sum_{(i_1 \cdots i_{m_n+1}) \in Q_{m_n}} r_{i_1 \cdots i_{m_n}}^s \\
&\geq K' \frac{N_{C_2 r_n}(F)}{-\log r_n} r_n^s \\
&> K' r_n^{s - \overline{\dim}_B F + 2\delta} \geq 1
\end{aligned}$$

for all sufficiently large n. By Theorem 4.5.1, we obtain $\overline{P}_Q(s\Phi) \ge 0$ for every $s \le \overline{\dim}_B F - 2\delta$. Since $\delta > 0$ is arbitrary, it follows from the second property in Theorem 4.2.2 that $\overline{P}_Q(\overline{\dim}_B F\Phi) \ge 0$, and hence, $\overline{\dim}_B F \le s_{\overline{P}}$.

Now we establish the last property in the theorem. Since the sequence Φ is subadditive and the set Q is compact and σ-invariant, it follows from the second property in Theorem 4.2.6 that

$$P_Q(s\Phi) = \underline{P}_Q(s\Phi) = \overline{P}_Q(s\Phi)$$

for each $s \in \mathbb{R}$. Hence, $s_P = s_{\underline{P}} = s_{\overline{P}}$. The desired statement follows now readily from the first property and Theorem 4.5.1. □

The following example illustrates that we may have strict inequalities in the first property of Theorem 8.1.3 when the sequence Φ is not subadditive.

Example 8.1.4. There is a generalized Moran construction in $\mathbb{R}$ modeled by Σ_2^+ such that:

1. each basic set $\Delta_{i_1 \cdots i_n}$ is a closed interval of length depending only on n;
2. the sequence Φ is not subadditive;
3. $s_P = s_{\underline{P}} = \dim_H F = \underline{\dim}_B F < \overline{\dim}_B F = s_{\overline{P}}$.

Construction. Let λ_n and n_k be the sequences of numbers constructed in Example 4.5.2. We consider a generalized Moran construction modeled by Σ_2^+ such that each basic set $\Delta_{i_1 \cdots i_n}$ is a closed interval of length $\exp \lambda_n$ (the location of each basic set is arbitrary). The second property follows from Example 4.5.2.

Now we establish the last property in the example. Clearly, $N_{\exp \lambda_n}(F) \le 2^n$, where $N_\delta(F)$ denotes the least number of sets of diameter at most δ needed to cover F. Given an interval of length $\exp \lambda_n$, there exist at most two basic sets intersecting it, which implies that

$$2N_{\exp \lambda_n}(F) \ge 2^n.$$

Since $\lambda_{n+1} - \lambda_n \ge -a$, the lower and upper box dimensions of F are given by

$$\begin{aligned} \underline{\dim}_B F &= \liminf_{n\to\infty} \frac{\log N_{\exp \lambda_n}(F)}{-\lambda_n} \\ &= \log 2 \times \liminf_{n\to\infty} \frac{n}{-\lambda_n} = \frac{\log 2}{a}, \end{aligned}$$

and

$$\begin{aligned} \overline{\dim}_B F &= \limsup_{n\to\infty} \frac{\log N_{\exp \lambda_n}(F)}{-\lambda_n} \\ &= \log 2 \times \limsup_{n\to\infty} \frac{n}{-\lambda_n} = \frac{\log 2}{b}. \end{aligned}$$

Since the geometric construction is a generalized Moran construction, it follows from Theorem 8.1.3 that $\dim_H F \geq s_P$. Since $\lambda_n \geq -an$ for each $n \in \mathbb{N}$, we obtain $s_P \geq \log 2/a$, and hence,

$$\log 2/a = s_P = \dim_H F = \underline{\dim}_B F < \overline{\dim}_B F = \log 2/b.$$

The construction in Example 4.5.2 shows that $s_{\underline{P}} = \log 2/a$ and $s_{\overline{P}} = \log 2/b$. □

8.2 Moran constructions

We consider in this section the special class of Moran constructions, in which the radii $r_{i_1 \cdots i_n}$ in Definition 8.1.2 are replaced by a product $\prod_{k=1}^n \lambda_{i_k}$, for some constants $\lambda_1, \ldots, \lambda_\kappa$. This causes the dimension formulas obtained in the former section to become much simpler.

We first recall the notion of Moran construction.

Definition 8.2.1. Moran constructions are geometric constructions such that:

1. for each $n, m \in \mathbb{N}$ and $(i_1 i_2 \cdots), (j_1 j_2 \cdots) \in Q$, the sets $\Delta_{i_1 \cdots i_n}$ and $\Delta_{j_1 \cdots j_m}$ are geometrically similar;
2. each basic set is the closure of its interior;
3. $\operatorname{int} \Delta_{i_1 \cdots i_n} \cap \operatorname{int} \Delta_{j_1 \cdots j_n} = \varnothing$ whenever $(i_1 \cdots i_n) \neq (j_1 \cdots j_n)$;
4. $\operatorname{diam} \Delta_{i_1 \cdots i_n} = \prod_{k=1}^n \lambda_{i_k}$ for some constants $\lambda_1, \ldots, \lambda_\kappa \in (0,1)$.

These constructions were introduced by Moran [137] in the case of the full shift. We refer to [155] for the discussion of the case of a general symbolic dynamics.

Now let F be the limit set of a Moran construction. In [155], Pesin and Weiss showed that

$$\dim_H F = \underline{\dim}_B F = \overline{\dim}_B F = t,$$

where t is the unique root of the equation $P_Q(t\varphi) = 0$ for the function $\varphi\colon Q \to \mathbb{R}$ defined by $\varphi(\omega) = \log \lambda_{i_1(\omega)}$. Here P_Q is the classical topological pressure with respect to the shift map $\sigma|Q$. This is also a consequence of the second property in Theorem 8.1.3, because a Moran construction is a generalized Moran construction for which the sequence Φ in Theorem 8.1.3 is additive.

Example 8.2.2. When $Q = \Sigma_\kappa^+$, we have

$$\begin{aligned} P_{\Sigma_\kappa^+}(t\varphi) &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{(i_1 \cdots i_n)} \prod_{j=1}^n \lambda_{i_j}^t \\ &= \lim_{n\to\infty} \frac{1}{n} \log \left[\left(\sum_{i=1}^\kappa \lambda_i^t \right)^n \right] = \log \sum_{i=1}^\kappa \lambda_i^t, \end{aligned}$$

and thus we can write Bowen's equation in the form $\sum_{i=1}^\kappa \lambda_i^t = 1$. This formula was first obtained by Moran in [137].

Example 8.2.3. When $Q = \Sigma_A^+$ is a topological Markov chain, one can write Bowen's equation in the form $\rho(A\Lambda_t) = 1$, where ρ is the spectral radius, and where Λ_t is the $\kappa \times \kappa$ diagonal matrix with entries $\lambda_1^t, \ldots, \lambda_\kappa^t$ in the diagonal (see [155]). We present an elementary derivation of this formula. For each $n \in \mathbb{N}$ and $t \geq 0$, let $S_n(t)$ be the sum of all entries of the matrix $\Lambda_t(A\Lambda_t)^n$. We have $S_n(t) \leq c_n \rho(A\Lambda_t)^n$, where c_n grows at most polynomially in n. Since A and Λ_t are nonnegative matrices, we have

$$S_n(t) \geq \operatorname{tr}(A\Lambda_t)^n \min_i \lambda_i^t.$$

For each $t \geq 0$, we thus obtain

$$\begin{aligned}\log \rho(A\Lambda_t) &= \lim_{n\to\infty} \frac{1}{n} \log S_n(t) \\ &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{(i_1\cdots i_n)} \lambda_{i_1}^t a_{i_1 i_2} \cdots a_{i_{n-1} i_n} \lambda_{i_n}^t = P_{\Sigma_A^+}(t\varphi).\end{aligned}$$

Hence, in this case Bowen's equation can be written in the form $\rho(A\Lambda_t) = 1$.

8.3 Expanding induced maps

We consider in this section the particular case when a geometric construction has associated an expanding induced map. This allows one to introduce the so-called ratio coefficients of the construction, which can then be used to obtain dimension estimates for the limit set.

Let F be the limit set of construction modeled by Q. We always require in this section that

$$\Delta_{i_1\cdots i_n} \cap \Delta_{j_1\cdots j_n} = \varnothing \quad \text{whenever} \quad (i_1 \cdots i_n) \neq (j_1 \cdots j_n),$$

for each $(i_1 i_2 \cdots), (j_1 j_2 \cdots) \in Q$ and $n \in \mathbb{N}$. This condition guarantees that the coding map $\chi\colon Q \to F$ given by (8.1) is a homeomorphism. We then define an *induced map* $g\colon F \to F$ in the limit set by

$$g = \chi \circ \sigma \circ \chi^{-1},$$

where $\sigma\colon Q \to Q$ is the shift map. We thus have the commutative diagram

$$\begin{array}{ccc} Q & \xrightarrow{\ \sigma\ } & Q \\ {\scriptstyle\chi}\big\downarrow & & \big\downarrow{\scriptstyle\chi} \\ F & \xrightarrow{\ g\ } & F \end{array} .$$

It is easy to verify that g is continuous and onto. Moreover, given $n \in \mathbb{N}$, the map g^n is invertible on each set $F \cap \Delta_{i_1\cdots i_n}$. Therefore, for each $\omega = (i_1 i_2 \cdots) \in Q$

and $n, k \in \mathbb{N}$, we can define the two families of numbers

$$\underline{\lambda}_k(\omega, n) = \min \inf \left\{ \frac{\|x - y\|}{\|g^n(x) - g^n(y)\|} : x, y \in F \cap \Delta_{j_1 \cdots j_{n+k}} \text{ and } x \neq y \right\}$$

and

$$\overline{\lambda}_k(\omega, n) = \max \sup \left\{ \frac{\|x - y\|}{\|g^n(x) - g^n(y)\|} : x, y \in F \cap \Delta_{j_1 \cdots j_{n+k}} \text{ and } x \neq y \right\},$$

where the minimum and maximum are taken over all Q-admissible finite sequences $(j_1 \cdots j_{n+k})$ such that $(j_1 \cdots j_n) = (i_1 \cdots i_n)$. These numbers are called respectively *lower* and *upper ratio coefficients* of the geometric construction.

It follows from Proposition 5.1.4 that if g is a continuous expanding map (see Definition 5.1.1), then

$$a^{-n} \leq \underline{\lambda}_k(\omega, n) \leq \overline{\lambda}_k(\omega, n) \leq b^{-n} \tag{8.3}$$

for each $\omega \in Q$, $n \in \mathbb{N}$, and all sufficiently large $k \in \mathbb{N}$. In particular, the map g is locally bi-Lipschitz. When g is not expanding, it may happen that $\underline{\lambda}_k(\omega, n) = 0$ or $\overline{\lambda}_k(\omega, n) = +\infty$ for some $\omega \in Q$ and $n \in \mathbb{N}$.

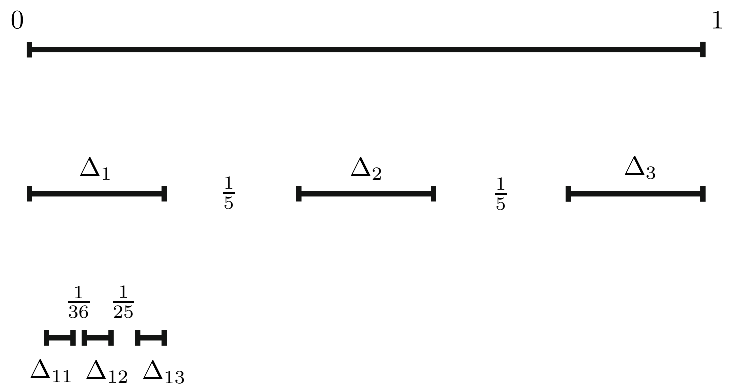

Figure 8.3: A geometric construction without expanding induced map

Example 8.3.1. There exists a geometric construction in $\mathbb{R}$ modeled by Σ_3^+ such that:

1. each basic set $\Delta_{i_1 \cdots i_n}$ is a closed interval;
2. the induced map g is not expanding;
3. there exists $\omega \in \Sigma_3^+$ such that $\underline{\lambda}_k(\omega, n) = 0$ and $\overline{\lambda}_k(\omega, n) = +\infty$ for every $n, k \in \mathbb{N}$.

Construction. Let us write $\Delta_{i_1\cdots i_n j} = [a_{jn}, b_{jn}]$ for each $(i_1 \cdots i_n) = (0 \cdots 0)$ and $j = 0, 1, 2$. We choose the constants a_{jn} and b_{jn} so that (see Figure 8.3)

1. $b_{0n}, a_{1n}, b_{1n}, a_{2n} \in F$ for each $n \in \mathbb{N}$,
2. $a_{1n} - b_{0n}$ is $e^{-a(n+1)}$ for n even, and is $e^{-b(n+1)}$ for n odd,
3. $a_{2n} - b_{1n}$ is $e^{-a(n+1)}$ for $\lfloor n/2 \rfloor$ even, and is $e^{-b(n+1)}$ for $\lfloor n/2 \rfloor$ odd,

where a and b are positive constants with $a \neq b$ (for example, in Figure 8.3 we have $a = \log 5$, $b = \log 6$, and the intervals $\Delta_{i_1\cdots i_n}$ have length 5^{-n}). This guarantees that $\underline{\lambda}_k(\omega, n) = 0$ and $\overline{\lambda}_k(\omega, n) = +\infty$ when $\omega = (00\cdots)$, for every $n, k \in \mathbb{N}$. □

Now we obtain several dimension estimates for the limit set of a geometric construction with an expanding induced map. We first consider a particular class of constructions.

Definition 8.3.2. A geometric construction is said to be a *Markov construction* if it is modeled by Σ_A^+ for some transition matrix A.

Now let F be the limit set of a Markov construction with a transitive transition matrix A. This means that there exists $q \in \mathbb{N}$ such that A^q has only positive entries. It is easy to verify that the induced map $g\colon F \to F$ is a local homeomorphism at every point. We assume that g is expanding, and we take $k \in \mathbb{N}$ such that the ratio coefficients satisfy (8.3) for every $\omega \in \Sigma_A^+$ and $n \in \mathbb{N}$. We then define two sequences of functions $\underline{\varphi}_{k,n}$ and $\overline{\varphi}_{k,n}$ in Σ_A^+ by

$$\underline{\varphi}_{k,n}(\omega) = \log \underline{\lambda}_k(\omega, n) \quad \text{and} \quad \overline{\varphi}_{k,n}(\omega) = \log \overline{\lambda}_k(\omega, n), \tag{8.4}$$

and we denote them respectively by $\underline{\Phi}_k$ and $\overline{\Phi}_k$. Since the functions in (8.4) are constant on each basic set $\Delta_{i_1\cdots i_n}$, the sequences $\underline{\Phi}_k$ and $\overline{\Phi}_k$ have tempered variation (see (4.2)). By (8.3) and the second property in Theorem 4.4.1, there exist unique roots $\underline{s}_k$ and $\overline{s}_k$ respectively of the equations

$$\overline{P}_{\Sigma_A^+}(s\underline{\Phi}_k) = 0 \quad \text{and} \quad P_{\Sigma_A^+}(s\overline{\Phi}_k) = 0.$$

The following statement is now a simple consequence of Theorem 5.1.7.

Theorem 8.3.3. *If F is the limit set of a Markov construction with expanding induced map, then*

$$\sup_{k\in\mathbb{N}} \underline{s}_k \leq \dim_H F \leq \underline{\dim}_B F \leq \overline{\dim}_B F \leq \inf_{k\in\mathbb{N}} \overline{s}_k.$$

Moreover, by Theorem 5.1.12, for each open set U such that $F \cap U \neq \varnothing$, we have

$$\dim_H(F \cap U) = \dim_H F, \ \underline{\dim}_B(F \cap U) = \underline{\dim}_B F, \ \overline{\dim}_B(F \cap U) = \overline{\dim}_B F.$$

Now we consider the class of asymptotically conformal induced maps (see Definition 5.1.13). The following is a simple consequence of Theorem 5.1.14.

Theorem 8.3.4. *Let F be the limit set of a Markov construction with expanding induced map which is asymptotically conformal on F. Then, for each open set U such that $F \cap U \neq \varnothing$, we have*

$$\dim_H(F \cap U) = \underline{\dim}_B(F \cap U) = \overline{\dim}_B(F \cap U) = s, \tag{8.5}$$

where s is the unique root of the equation

$$\lim_{n\to\infty} \frac{1}{n} \log \sum_{(i_1\cdots i_n)\in S_n} \big(\operatorname{diam}(F \cap \Delta_{i_1\cdots i_n})\big)^s = 0,$$

and where S_n is the set of all Σ_A^+-admissible sequences of length n.

The following example shows that (8.5) may not hold when g is not asymptotically conformal, even if each basic set is an interval.

Example 8.3.5. There exists a geometric construction in $\mathbb{R}$ modeled by Σ_2^+ such that:

1. each basic set $\Delta_{i_1\cdots i_n}$ is a closed interval of length depending only on n;
2. the map g is expanding, but is not asymptotically conformal;
3. $\dim_H F = \underline{\dim}_B F < \overline{\dim}_B F$.

Construction. We start with a lemma.

Lemma 8.3.6. *Let us assume that g is a local homeomorphism at every point, and that there exist constants $a \geq b > 1$ such that*

$$a^{-1} \leq \underline{\lambda}_k(\omega, 1) \leq \overline{\lambda}_k(\omega, 1) < b^{-1}$$

for some integer $k \in \mathbb{N}$ and all $\omega \in Q$. Then the map g is expanding.

Proof of the lemma. Since the set F is compact, there exists $r_0 > 0$ such that the domain of each local inverse of g contains a ball of radius r_0, and $g(B(x, r_0)) \supset B(g(x), br_0)$ for every $x \in X$. For each $x, y \in F$ such that $\|x - y\| \leq r_0$, we have

$$b\|x - y\| \leq \|g(x) - g(y)\| \leq a\|x - y\|.$$

For $x \in F$ and $0 < r < r_0$, we obtain

$$\begin{aligned} g(B(x, r)) &= \big\{g(y) : \|x - y\| < r\big\} \\ &\subset \big\{g(y) : \|g(x) - g(y)\| < ar\big\} = B(g(x), ar). \end{aligned}$$

Now let $z \in B(g(x), br)$ and set $y = g(z)$. We have

$$b\|x - y\| \leq \|g(x) - g(y)\| \leq br,$$

and hence $\|x - y\| < r$. This shows that $z = g(y) \in g(B(x, r))$, and the map g is expanding. ∎

Now let λ_n be the sequence constructed in Example 4.5.2. We describe a geometric construction modeled by Σ_2^+ such that each basic set $\Delta_{i_1\cdots i_n}$ is a closed interval of length $\exp\lambda_n$. We show that the location of the basic sets can be chosen in such a way that the induced map g on the limit set is expanding. Given $\Delta_{i_1\cdots i_n}$, we require that $\Delta_{i_1\cdots i_n 0}$ and $\Delta_{i_1\cdots i_n 1}$ start respectively at the left and right endpoints of the interval $\Delta_{i_1\cdots i_n}$. See Figure 8.4, where the ratio

$$\operatorname{diam}\Delta_{i_1\cdots i_{n+1}}/\operatorname{diam}\Delta_{i_1\cdots i_n}=\exp(\lambda_{n+1}-\lambda_n)$$

is 4^{-1} or 5^{-1}, that is, $a=\log 5$ and $b=\log 4$ in Example 4.5.2. The number in each arrow is $\operatorname{diam}\Delta_{i_1\cdots i_{n+1}}/\operatorname{diam}\Delta_{i_1\cdots i_n}$. For each $x,y\in F\cap\Delta_{i_1\cdots i_n}$ with $x\neq y$, we have

$$\frac{\|x-y\|}{\|g^n(x)-g^n(y)\|}=\frac{e^{\lambda_m}+\sum_{j=1}^\infty k_j e^{\lambda_{m+j}}}{e^{\lambda_{m-n}}+\sum_{j=1}^\infty k_j e^{\lambda_{m-n+j}}}$$

for some $m>n$, and $k_j\in\{-2,-2,0,1,2\}$ for each $j\in\mathbb{N}$. We notice that not all sequences $(k_j)_{j\in\mathbb{N}}$ are admissible. By the construction in Example 4.5.2, we have

$$-an\leq\lambda_m-\lambda_{m-n}\leq -bm.$$

Therefore,

$$\begin{aligned}\overline{\lambda}_k(\omega,n)&\leq\sup_{m\in\mathbb{N}}\frac{e^{\lambda_m}}{e^{\lambda_{m-n}}}\times\frac{1+2\sum_{j=1}^\infty e^{-bj}}{1-2\sum_{j=1}^\infty e^{-bj}}\\&\leq\frac{e^{-bn}(1+e^{-b})}{1-2e^{-b}}<\infty,\end{aligned}$$

and

$$\begin{aligned}\underline{\lambda}_k(\omega,n)&\geq\inf_{m\in\mathbb{N}}\frac{e^{\lambda_m}}{e^{\lambda_{m-n}}}\times\frac{1-2\sum_{j=1}^\infty e^{-bj}}{1+2\sum_{j=1}^\infty e^{-bj}}\\&\geq\frac{e^{-an}(1-3e^{-b})}{1+e^{-b}}>0,\end{aligned}$$

for all $a\geq b>\log 3$. Provided that b is sufficiently large, we have

$$0<\frac{e^{-a}(1-3e^{-b})}{1+e^{-b}}\leq\frac{e^{-b}(1+e^{-b})}{1-3e^{-b}}<1.$$

By Lemma 8.3.6, we conclude that the map g is expanding. On the other hand, by the construction of the sequence λ_n, we have

$$\sup\left\{\lambda_m-\lambda_{m-n}:m\in\mathbb{N}\right\}=-bn$$

and

$$\inf\left\{\lambda_m-\lambda_{m-n}:m\in\mathbb{N}\right\}=-an$$

for each $n \in \mathbb{N}$. Thus,

$$\overline{\lambda}_k(\omega, n) \geq \sup_{m\in\mathbb{N}} \frac{e^{\lambda_m}}{e^{\lambda_{m-n}}} \times \frac{1 - 2\sum_{j=1}^{\infty} e^{-bj}}{1 + 2\sum_{j=1}^{\infty} e^{-bj}} \geq \frac{e^{-bn}(1 - 3e^{-b})}{1 + e^{-b}},$$

and

$$\underline{\lambda}_k(\omega, n) \leq \inf_{m\in\mathbb{N}} \frac{e^{\lambda_m}}{e^{\lambda_{m-n}}} \times \frac{1 + 2\sum_{j=1}^{\infty} e^{-bj}}{1 - 2\sum_{j=1}^{\infty} e^{-bj}} \leq \frac{e^{-an}(1 + e^{-b})}{1 - 3e^{-b}}.$$

This shows that the map g is not asymptotically conformal. Finally, the last property in Example 8.3.5 follows from the construction in Example 8.1.4. ☐

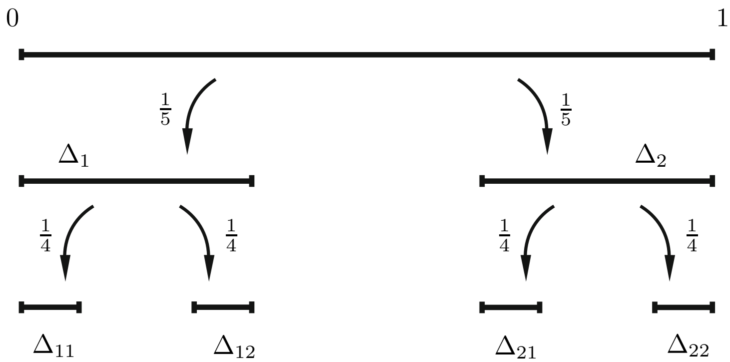

Figure 8.4: A construction without asymptotically conformal induced map

Let again F be the limit set of a geometric construction. By a result of Parry in [148], the induced map $g\colon F \to F$ is a local homeomorphism at every point if and only if $\sigma|Q$ is topologically conjugate to a topological Markov chain. Hence, in general the map g may not be expanding. However, a slight modification of the proof of Theorem 5.1.7 establishes the following result.

Theorem 8.3.7. *Let F be the limit set of a geometric construction such that the induced map g satisfies*

$$B(g(x), br) \subset g(B(x, r)) \subset B(g(x), ar)$$

for every $x \in F$ and $0 < r < r_0$, where $a \geq b > 1$ and $r_0 > 0$ are constants. Then

$$\overline{\dim}_B F \leq \inf_{k\in\mathbb{N}} \overline{s}_k.$$

The proof of Proposition 5.1.4 shows that with the hypotheses of Theorem 8.3.7, the inequalities in (8.3) still hold. Hence, for all sufficiently large $k \in \mathbb{N}$, the number $\overline{s}_k$ is well defined.

We also consider briefly the case of geometric constructions obtained from contraction maps. More precisely, these are constructions defined by κ contraction maps $f_i\colon \Delta \to \Delta$ on some closed set $\Delta \subset \mathbb{R}^m$, for $i = 1, \dots, \kappa$. We require that the closed sets $\Delta_i = f_i(\Delta)$ are disjoint. The basic sets of the construction are defined by

$$\Delta_{i_1\cdots i_n} = (f_{i_1} \circ \cdots \circ f_{i_n})(\Delta).$$

Moreover, the map $g\colon F \to F$ satisfies $g(x) = f_i^{-1}(x)$ for each $x \in F \cap \Delta_i$ and $i = 1, \dots, \kappa$. Let $f_{i_1\cdots i_n} = f_{i_1} \circ \cdots \circ f_{i_n}$. We define numbers

$$0 \le \underline{\mu}_{i_1\cdots i_n} \le \overline{\mu}_{i_1\cdots i_n} < 1$$

for each $(i_1 i_2 \cdots) \in Q$ and $n \in \mathbb{N}$ by

$$\underline{\mu}_{i_1\cdots i_n} = \inf\left\{\frac{\|f_{i_1\cdots i_n}(x) - f_{i_1\cdots i_n}(y)\|}{\|x-y\|} : x, y \in \Delta \text{ and } x \ne y\right\},$$

and

$$\overline{\mu}_{i_1\cdots i_n} = \sup\left\{\frac{\|f_{i_1\cdots i_n}(x) - f_{i_1\cdots i_n}(y)\|}{\|x-y\|} : x, y \in \Delta \text{ and } x \ne y\right\}.$$

We observe that

$$\underline{\mu}_i\|x-y\| \le \|f_i(x) - f_i(y)\| \le \overline{\mu}_i\|x-y\|$$

for every $x, y \in \Delta$. It is easy to verify that

$$\prod_{k=1}^n \underline{\mu}_{i_k} \le \underline{\mu}_{i_1\cdots i_n} \le \underline{\lambda}_k(\omega, n) \le \overline{\lambda}_k(\omega, n) \le \overline{\mu}_{i_1\cdots i_n} \le \prod_{k=1}^n \overline{\mu}_{i_k} \tag{8.6}$$

for every $\omega = (i_1 i_2 \cdots) \in Q$ and $n, k \in \mathbb{N}$.

Now we consider a Markov construction with contraction maps, with a transitive transition matrix A. If $\underline{\mu}_i > 0$ for $i = 1, \dots, \kappa$, then the induced map g is expanding, and it follows from Theorem 8.3.3 that

$$\sup_{k\in\mathbb{N}} \underline{s}_k \le \dim_H F \le \underline{\dim}_B F \le \overline{\dim}_B F \le \inf_{k\in\mathbb{N}} \overline{s}_k.$$

We may also use the numbers $\underline{\mu}_i$ and $\overline{\mu}_i$ to estimate the dimension, but in general this may give worse estimates. Namely, it follows from (8.6) and the first property in Theorem 4.2.2 that

$$\underline{t} \le \dim_H F \le \underline{\dim}_B F \le \overline{\dim}_B F \le \overline{t},$$

where t and $\overline{t}$ are the unique roots respectively of the equations

$$P_{\Sigma_A^+}(t\underline{\varphi}) = 0 \quad \text{and} \quad P_{\Sigma_A^+}(t\overline{\varphi}) = 0$$

for the functions $\underline{\varphi}$ and $\overline{\varphi}$ defined in Σ_A^+ by

$$\underline{\varphi}(\omega) = \log \underline{\mu}_{i_1}(\omega) \quad \text{and} \quad \overline{\varphi}(\omega) = \log \overline{\mu}_{i_1}(\omega).$$

We have $\underline{t} \le \underline{s}_k$ and $\overline{s}_k \le \overline{t}$, and these inequalities can be strict.

Chapter 9

Entropy Spectra

For the class of asymptotically subadditive sequences, we describe in this chapter a multifractal analysis of the entropy spectra of their generalized Birkhoff averages. More precisely, for an asymptotically subadditive sequence $\Phi = (\varphi_n)_{n\in\mathbb{N}}$ we consider the level sets $E(\alpha)$ composed of the points x such that $\varphi_n(x)/n \to \alpha$ when $n \to \infty$. The associated entropy spectrum $\mathcal{E}$ is obtained from computing the topological entropy of the level sets $E(\alpha)$ as a function of α, and its multifractal analysis corresponds to describing the properties of the function $\mathcal{E}$ in terms of the thermodynamic formalism. For this we use the thermodynamic formalism for asymptotically subadditive sequences developed in Chapter 7. We consider the general case when the Kolmogorov–Sinai entropy is not upper semicontinuous and when the topological pressure is not differentiable. We also consider multidimensional sequences, that is, vectors of asymptotically subadditive sequences.

9.1 Preliminaries

We introduce in this section the generalized Birkhoff averages that are used to define the entropy spectra. We also establish some of their basic properties.

Let $f\colon X \to X$ be a continuous transformation of a compact metric space. For an asymptotically subadditive sequence Φ of continuous functions $\varphi_n\colon X \to \mathbb{R}$, we define

$$\lambda(x) = \lim_{n\to\infty} \frac{\varphi_n(x)}{n}, \tag{9.1}$$

whenever the limit exists. This number was considered by Feng and Huang in [66].

Theorem 9.1.1 ([66]). *If Φ is an asymptotically subadditive sequence and μ is an f-invariant probability measure in X, then:*

1. *the limit $\lambda(x)$ exists for μ-almost every $x \in X$, and there exists a constant $C > 0$ (independent of μ) such that $\lambda(x) \le C$ for μ-almost every $x \in X$;*

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Progress in Mathematics 294, DOI 10.1007/978-3-0348-0206-2_9, © Springer Basel AG 2011

2. $\int_X \lambda(x)\,d\mu(x) = F(\mu)$, *where*

$$F(\mu) = \lim_{n\to\infty} \frac{1}{n} \int_X \varphi_n \, d\mu, \tag{9.2}$$

and if μ is ergodic, then $\lambda(x) = F(\mu)$ for μ-almost every $x \in X$;

3. *if $\mu = \int_{\mathcal{M}_f} \nu \, d\tau(\nu)$ is an ergodic decomposition of μ, then*

$$F(\mu) = \int_{\mathcal{M}_f} F(\nu)\, d\tau(\nu). \tag{9.3}$$

Proof. Given $\varepsilon > 0$, there exists a subadditive sequence $(\psi_n)_{n\in\mathbb{N}}$ and $n_0 \in \mathbb{N}$ such that

$$\|\varphi_n - \psi_n\|_\infty < n\varepsilon \quad \text{for} \quad n \geq n_0. \tag{9.4}$$

Therefore, by Kingman's subadditive ergodic theorem, we have

$$\limsup_{n\to\infty} \frac{\varphi_n(x)}{n} \leq \lim_{n\to\infty} \frac{\psi_n(x)}{n} + \varepsilon \leq \liminf_{n\to\infty} \frac{\varphi_n(x)}{n} + 2\varepsilon \tag{9.5}$$

for μ-almost every $x \in X$. Since ε is arbitrary, this implies that $\lambda(x)$ is well defined for μ-almost every $x \in X$. Now we establish the existence of a universal constant C. Set $D = \max_{x\in X} \psi_{n_0}(x)$. Then $\psi_{kn_0} \leq kD$, by the subadditivity of the sequence, and we obtain

$$\lambda(x) \leq \limsup_{k\to\infty} \frac{\psi_{kn_0}(x)}{kn_0} + \varepsilon \leq \frac{D}{n_0} + \varepsilon$$

for μ-almost every $x \in X$.

Now we establish the second property in the theorem. We recall that the existence of the limit $F(\mu)$ in (9.2) was already established in the proof of Theorem 7.2.1. Given $\varepsilon > 0$, we consider the same subadditive sequence $(\psi_n)_{n\in\mathbb{N}}$ in (9.4). Again by Kingman's subadditive ergodic theorem, we have

$$\int_X \lim_{n\to\infty} \frac{\psi_n(x)}{n}\, d\mu(x) = \lim_{n\to\infty} \frac{1}{n} \int_X \psi_n \, d\mu.$$

It thus follows from (9.5) that

$$\int_X \lambda(x)\, d\mu(x) \leq \lim_{n\to\infty} \frac{1}{n} \int_X \psi_n \, d\mu + \varepsilon \leq \int_X \lambda(x)\, d\mu(x) + 2\varepsilon. \tag{9.6}$$

Since

$$F(\mu) \leq \lim_{n\to\infty} \frac{1}{n} \int_X \psi_n \, d\mu + \varepsilon \leq F(\mu) + 2\varepsilon, \tag{9.7}$$

by (9.6) we have

$$\int_X \lambda(x)\, d\mu(x) \leq F(\mu) + 2\varepsilon \leq \int_X \lambda(x)\, d\mu(x) + 4\varepsilon.$$

Since ε is arbitrary, we conclude that

$$\int_X \lambda(x)\, d\mu(x) = F(\mu).$$

When μ is ergodic, it follows again from Kingman's subadditive ergodic theorem that

$$\lim_{n\to\infty} \frac{\psi_n(x)}{n} = \lim_{n\to\infty} \frac{1}{n} \int_X \psi_n\, d\mu$$

for μ-almost every $x \in X$. Therefore, by (9.5) and (9.7),

$$\begin{aligned}\lambda(x) &\le \lim_{n\to\infty} \frac{1}{n} \int_X \psi_n\, d\mu + \varepsilon \le F(\mu) + 2\varepsilon \\ &\le \lim_{n\to\infty} \frac{1}{n} \int_X \psi_n\, d\mu + 2\varepsilon \le \lambda(x) + 3\varepsilon\end{aligned}$$

for μ-almost every $x \in X$. Since ε is arbitrary, we conclude that $\lambda(x) = F(\mu)$ for μ-almost every $x \in X$.

For the last property, we first assume that the sequence Φ is subadditive. Then

$$\frac{1}{n} \int_X \varphi_n\, d\nu \le \|\varphi_1\|_\infty$$

for every $n \in \mathbb{N}$ and $\nu \in \mathcal{M}_f$. For each $k \in \mathbb{N}$ and $\nu \in \mathcal{M}_f$, we define

$$g_k(\nu) = \frac{1}{2^k} \int_X \varphi_{2^k}\, d\nu.$$

Since Φ is subadditive and ν is f-invariant, we have

$$\|\varphi_1\|_\infty \ge g_1(\nu) \ge g_2(\nu) \ge \cdots,$$

and $g_k(\nu) \searrow F(\nu)$ when $k \to \infty$. Therefore,

$$\begin{aligned}F(\mu) &= \lim_{n\to\infty} \frac{1}{n} \int_X \varphi_n\, d\mu \\ &= \lim_{n\to\infty} \int_{\mathcal{M}_f} \frac{1}{n} \int_X \varphi_n\, d\nu\, d\tau(\nu) \\ &= \lim_{k\to\infty} \int_{\mathcal{M}_f} \frac{1}{2^k} \int_X \varphi_{2^k}\, d\nu\, d\tau(\nu) \\ &= \lim_{k\to\infty} \int_{\mathcal{M}_f} g_k(\nu)\, d\tau(\nu) \\ &= \int_{\mathcal{M}_f} \lim_{k\to\infty} g_k(\nu)\, d\tau(\nu) = \int_{\mathcal{M}_f} F(\nu)\, d\tau(\nu),\end{aligned}$$

using the monotone convergence theorem in the fifth identity. This establishes identity (9.3) when the sequence Φ is subadditive.

Now we assume that Φ is asymptotically subadditive. Given $\varepsilon > 0$, let again $(\psi_n)_{n\in\mathbb{N}}$ be a subadditive sequence satisfying (9.4). Then

$$\left|F(\nu) - \lim_{n\to\infty}\frac{1}{n}\int_X \psi_n\,d\nu\right| = \left|\lim_{n\to\infty}\frac{1}{n}\int_X \varphi_n\,d\nu - \lim_{n\to\infty}\frac{1}{n}\int_X \psi_n\,d\nu\right| \le \varepsilon \tag{9.8}$$

for every $\nu \in \mathcal{M}_f$. By the first part of the argument (for a subadditive sequence), we have

$$\int_{\mathcal{M}_f} \lim_{n\to\infty}\frac{1}{n}\int_X \psi_n\,d\nu\,d\tau(\nu) = \lim_{n\to\infty}\frac{1}{n}\int_X \psi_n\,d\mu.$$

It thus follows from (9.8) that

$$\begin{aligned}
&\left|\int_{\mathcal{M}_f} F(\nu)\,d\tau(\nu) - \lim_{n\to\infty}\frac{1}{n}\int_X \psi_n\,d\mu\right| \\
&= \left|\int_{\mathcal{M}_f}\left(F(\nu) - \lim_{n\to\infty}\frac{1}{n}\int_X \psi_n\,d\nu\right)d\tau(\nu)\right| \\
&\le \int_{\mathcal{M}_f}\left|F(\nu) - \lim_{n\to\infty}\frac{1}{n}\int_X \psi_n\,d\nu\right|\,d\tau(\nu) \le \varepsilon,
\end{aligned}$$

and hence,

$$\left|F(\mu) - \int_{\mathcal{M}_f} F(\nu)\,d\tau(\nu)\right| \le \left|\lim_{n\to\infty}\frac{1}{n}\int_X \varphi_n\,d\mu - \lim_{n\to\infty}\frac{1}{n}\int_X \psi_n\,d\mu\right| + \varepsilon \le 2\varepsilon.$$

Since ε is arbitrary, we obtain identity (9.3). This completes the proof of the theorem. ☐

9.2 Entropy spectra

We introduce in this section the notion of an entropy spectrum, obtained from the topological entropy of the level sets of the function λ in (9.1). We also describe a multifractal analysis of these spectra that shall be obtained later as a consequence of a more general result for multidimensional sequences.

To define the entropy spectra we consider the level sets of the function λ. Namely, for each $\alpha \in \mathbb{R}$ we define

$$E(\alpha) = \{x \in X : \lambda(x) = \alpha\}.$$

Following the general concept of multifractal analysis proposed in [16], we define the *entropy spectrum* of the function λ by

$$\mathcal{E}(\alpha) = h(f|E(\alpha)), \tag{9.9}$$

using the notion of topological entropy for arbitrary subsets of a compact metric space (see the discussion at the end of Section 4.1). We also consider the quantity

$$B(\Phi) = \limsup_{n\to\infty} \frac{1}{n} \sup_{x\in X} \varphi_n(x).$$

Proposition 9.2.1 ([66]). *If Φ is an asymptotically subadditive sequence, then*

$$\begin{aligned} B(\Phi) &= \lim_{n\to\infty} \frac{1}{n} \sup_{x\in X} \varphi_n(x) \\ &= \max\left\{F(\mu) : \mu \in \mathcal{M}_f\right\} \in \mathbb{R} \cup \{-\infty\}. \end{aligned}$$

Moreover, there exists an ergodic measure $\nu \in \mathcal{M}_f$ such that $B(\Phi) = F(\nu)$.

Proof. Given $\varepsilon > 0$, let $(\psi_n)_{n\in\mathbb{N}}$ be a subadditive sequence satisfying (9.4). Setting $C = \|\psi_1\|_\infty$, we have $\varphi_n \le n(C+\varepsilon)$, and hence, $\|\varphi_n\|_\infty/n \le C+\varepsilon$. This implies that $B(\Phi) \in \mathbb{R} \cup \{-\infty\}$.

Now let us set $b_n = \sup_{x\in X} \psi_n(x)$. Since the sequence $(\psi_n)_{n\in\mathbb{N}}$ is subadditive, we have

$$b_{n+m} \le b_n + b_m \quad \text{for every} \quad m, n \in \mathbb{N},$$

and hence,

$$B(\Phi) \le \lim_{n\to\infty} \frac{b_n}{n} + \varepsilon \le \liminf_{n\to\infty} \frac{1}{n} \sup_{x\in X} \varphi_n(x) + 2\varepsilon.$$

Since ε is arbitrary, we conclude that

$$B(\Phi) = \lim_{n\to\infty} \frac{1}{n} \sup_{x\in X} \varphi_n(x).$$

Now take $\mu \in \mathcal{M}_f$. By (9.2), we have

$$F(\mu) \le \limsup_{n\to\infty} \sup_{x\in X} \frac{\varphi_n(x)}{n} = B(\Phi).$$

Hence,

$$\sup\left\{F(\mu) : \mu \in \mathcal{M}_f\right\} \le B(\Phi).$$

For the reverse inequality, we consider sequences $(x_n)_{n\in\mathbb{N}} \subset X$ and $(m_n)_{n\in\mathbb{N}} \subset \mathbb{N}$ with $m_n \nearrow \infty$ when $n \to \infty$, such that

$$B(\Phi) = \lim_{n\to\infty} \frac{\varphi_{m_n}(x_n)}{m_n}. \tag{9.10}$$

We also consider the sequence of measures

$$\mu_n = \frac{1}{m_n} \sum_{j=0}^{m_n-1} f_*^j \delta_{x_n} = \frac{1}{m_n} \sum_{j=0}^{m_n-1} \delta_{f^j(x_n)}.$$

Without loss of generality, by eventually rechoosing the sequence $(m_n)_{n\in\mathbb{N}}$, we assume in addition that the sequence $(\mu_n)_{n\in\mathbb{N}}$ converges to some probability measure μ in X. One can easily show that $\mu \in \mathcal{M}_f$. Moreover, it follows from (9.10) and Lemma 7.2.3 (now applied to the measures $\nu_n = \delta_{x_n}$) that

$$B(\Phi) = \lim_{n\to\infty} \frac{1}{m_n} \int_X \varphi_{m_n}\, d\delta_{x_n} \le F(\mu).$$

This shows that

$$B(\Phi) = \sup\bigl\{F(\mu) : \mu \in \mathcal{M}_f\bigr\}. \tag{9.11}$$

Finally, by Theorem 9.1.1, there exists an ergodic measure $\nu \in \mathcal{M}_f$ such that $B(\Phi) \le F(\nu)$. It follows from (9.11) that $B(\Phi) = F(\nu)$, and thus,

$$\begin{aligned} B(\Phi) &= \max\bigl\{F(\mu) : \mu \in \mathcal{M}_f\bigr\} \\ &= \max\bigl\{F(\mu) : \mu \in \mathcal{M}_f \text{ ergodic}\bigr\}. \end{aligned}$$

This completes the proof of the proposition. □

The following result of Feng and Huang in [66] is a multifractal analysis of the entropy spectrum. Given an asymptotically subadditive sequence Φ, we define a function $p\colon \mathbb{R}^+ \to \mathbb{R}$ by

$$p(q) = P(q\Phi). \tag{9.12}$$

Theorem 9.2.2. *Let $f\colon X \to X$ be a continuous transformation of a compact metric space with $h(f) < \infty$ such that the entropy map $\mu \mapsto h_\mu(f)$ is upper semicontinuous. If Φ is an asymptotically subadditive sequence with $B(\Phi) \ne -\infty$, then the following properties hold:*

1. *for each $t > 0$, if $\alpha = p'(t^+)$ or $\alpha = p'(t^-)$, then $E(\alpha) \ne \varnothing$ and*

$$\mathcal{E}(\alpha) = \inf\bigl\{P(q\Phi) - \alpha q : q > 0\bigr\} = P(q\Phi) - \alpha t;$$

2. *for each $\alpha \in \bigcup_{t>0}[p'(t^-), B(\Phi)]$ we have*

$$\inf\bigl\{P(q\Phi) - \alpha q : q > 0\bigr\} = \max\bigl\{h_\mu(f) : \mu \in \mathcal{M}_f, F(\mu) = \alpha\bigr\};$$

3. *if for some $t > 0$ the sequence $t\Phi$ has a unique equilibrium measure μ_t, then μ_t is ergodic,*

$$p'(t) = F(\mu_t), \quad E(p'(t)) \ne \varnothing, \quad \text{and} \quad \mathcal{E}(p'(t)) = h_{\mu_t}(f).$$

We shall obtain Theorem 9.2.2 as a consequence of Theorem 9.4.1, which considers the more general case of multidimensional sequences. We note that the first property in Theorem 9.2.2 does not require the differentiability of the function p at the point t.

9.3 General maps

We consider in this section the general case of maps for which the entropy is not necessarily upper semicontinuous. The following can be considered a preliminary result towards the multifractal analysis of entropy spectra in Theorem 9.2.2. We continue to consider the function p in (9.12).

Theorem 9.3.1 ([66]). *Let $f\colon X \to X$ be a continuous transformation of a compact metric space with $h(f) < \infty$. If Φ is an asymptotically subadditive sequence with $B(\Phi) \neq -\infty$, then the following properties hold:*

1. *the function p is continuous and convex, and $p(q)/q \to B(\Phi)$ when $q \to +\infty$;*
2. *for each $t > 0$, if $\alpha = p'(t^+)$ or $\alpha = p'(t^-)$, then setting*
$$E_\varepsilon(\alpha) = \{E(\beta) : |\beta - \alpha| < \varepsilon\}$$
we have
$$\lim_{\varepsilon \to 0} h(f|E_\varepsilon(\alpha)) = \inf\{P(q\Phi) - \alpha q : q > 0\} = P(t\Phi) - \alpha t, \tag{9.13}$$
with the first identity also satisfied for $\alpha = B(\Phi)$;
3. *for each $\alpha \in \bigcup_{t>0}[p'(t^-), p'(t^+)]$ we have*
$$\inf\{P(q\Phi) - \alpha q : q > 0\} = \lim_{\varepsilon \to 0} \sup\{h_\mu(f) : \mu \in \mathcal{M}_f, |F(\mu) - \alpha| < \varepsilon\},$$
with the first identity also satisfied for $\alpha = B(\Phi)$;
4. *for each $\alpha \in (\lim_{t \to 0^+} p'(t^-), B(\Phi))$ we have*
$$\inf\{P(q\Phi) - \alpha q : q > 0\} = \sup\{h_\mu(f) : \mu \in \mathcal{M}_f, F(\mu) = \alpha\}.$$

We shall obtain Theorem 9.3.1 as a consequence of Theorem 9.3.2, which considers the more general case of multidimensional sequences. We refer to [66] for an example of a continuous map without upper semicontinuous entropy showing that, even for additive sequences, in general the infimum in (9.13) may be strictly larger than $\mathcal{E}(\alpha) = f(f|E(\alpha))$ for each $\alpha \in [\underline{B}(\Phi), B(\Phi)]$, where
$$\underline{B}(\Phi) = \lim_{n \to \infty} \frac{1}{n} \inf_{x \in X} \varphi_n(x).$$
This example builds on a construction of Krieger in [117] of uniquely ergodic transformations on Cantor sets, although with a different purpose in mind.

Now we recall some basic notions from convex analysis (we refer to [163] for further details). Let U be an open convex subset of $\mathbb{R}^k$ and let $f\colon U \to \mathbb{R}$ be a continuous convex function. We recall that a function f is said to be *convex* if
$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

for every $x, y \in U$ and $t \in [0,1]$. Given $x \in U$, a vector $a \in \mathbb{R}^k$ is said to be a *subgradient* of f at x if

$$f(y) - f(x) \geq \langle a, y - x\rangle$$

for every $y \in U$, where $\langle \cdot, \cdot \rangle$ is the standard inner product. The set of all subgradients at x is called the *subdifferential* of f at x, and we denote it by $\partial_x f$. We note that $\partial_x f$ is a nonempty convex compact set for every $x \in X$. We also consider the set $\operatorname{ext}(\partial_x f)$ of the extreme points of $\partial_x f$ (see Section 7.4 for the definition of extreme point). When $\operatorname{ext}(\partial_x f) = \{a\}$, we say that f is *differentiable* at x and we write $f'(x) = a$. Moreover, the *relative interior* of a convex set C is the set of points $x \in C$ such that for each $y \in C$ there exist $z \in C$ and $t \in (0,1)$ for which $x = ty + (1-t)z$, and we denote it by $\operatorname{rel\,int} C$.

To formulate a multidimensional version of Theorem 9.3.1, we need to introduce some notation. Let

$$\Phi_1 = (\varphi_{1,n})_{n\in\mathbb{N}}, \ldots, \Phi_\kappa = (\varphi_{\kappa,n})_{n\in\mathbb{N}}$$

be asymptotically subadditive sequences. We write $\Phi = (\Phi_1, \ldots, \Phi_\kappa)$. Given $\alpha = (\alpha_1, \ldots, \alpha_\kappa) \in \mathbb{R}^\kappa$, we consider the level set

$$E(\alpha) = \big\{x \in X : \lambda_i(x) = \alpha_i \text{ for } i = 1, \ldots, \kappa\big\},$$

where

$$\lambda_i(x) = \lim_{n\to\infty} \frac{\varphi_{i,n}(x)}{n}, \tag{9.14}$$

whenever the limit exists. We define the *entropy spectrum* of the function $\lambda = (\lambda_1, \ldots, \lambda_\kappa)$ again by (9.9). We also define a function $p\colon \mathbb{R}^\kappa \to \mathbb{R}$ by

$$p(q) = P(\langle q, \Phi\rangle) = P\bigg(\sum_{i=1}^{\kappa} q_i \Phi_i\bigg) \tag{9.15}$$

for each $q = (q_1, \ldots, q_\kappa) \in \mathbb{R}^\kappa$. Given $x = (x_1, \ldots, x_\kappa)$ and $y = (y_1, \ldots, y_\kappa)$ in $\mathbb{R}^\kappa$, we write $x \geq y$ if $x_i \geq y_i$ for $i = 1, \ldots, \kappa$. Moreover, given $A \subset \mathbb{R}^\kappa$, we denote by $\Delta(A)$ the set of points $x \in \mathbb{R}^\kappa$ such that there exists a sequence $(y_n)_{n\in\mathbb{N}} \subset A$ converging to x with $x \geq y_n$ for every $n \in \mathbb{N}$. Finally, for simplicity of the notation we write $\mathbb{R}^\kappa_+$ for $(\mathbb{R}^+)^\kappa$.

Theorem 9.3.2 ([66]). *Let $f\colon X \to X$ be a continuous transformation of a compact metric space with $h(f) < \infty$. If $\Phi_1, \ldots, \Phi_\kappa$ are asymptotically subadditive sequences with*

$$\limsup_{n\to\infty} \frac{1}{n} \sup_{x\in X} \sum_{i=1}^{\kappa} \varphi_{i,n}(x) \neq -\infty, \tag{9.16}$$

then:

1. *the function p is continuous and convex in $\mathbb{R}^\kappa_+$;*

2. *for each* $t \in \mathbb{R}_+^\kappa$, *if* $\alpha \in \operatorname{ext}(\partial_t p)$, *then setting*

$$E_\varepsilon(\alpha) = \{E(\beta) : \|\beta - \alpha\| < \varepsilon\},$$

we have

$$\begin{aligned}\lim_{\varepsilon\to 0} h(f|E_\varepsilon(\alpha)) &= \inf\{P(\langle q, \Phi\rangle) - \langle \alpha, q\rangle : q \in \mathbb{R}_+^\kappa\} \\ &= P(\langle t, \Phi\rangle) - \langle \alpha, t\rangle,\end{aligned}$$

with the first identity also satisfied for $\alpha \in \Delta(\{\operatorname{ext}(\partial_q p) : q \in \mathbb{R}_+^\kappa\})$;

3. *for each* $\alpha \in \{\partial_q p : q \in \mathbb{R}_+^\kappa\}$ *and setting*

$$F(\mu) = \lim_{n\to\infty} \frac{1}{n}\left(\int_X \varphi_{1,n}\, d\mu, \ldots, \int_X \varphi_{\kappa,n}\, d\mu\right),$$

we have

$$\lim_{\varepsilon\to 0} \sup\{h_\mu(f) : \mu \in \mathcal{M}_f, \|F(\mu) - \alpha\| < \varepsilon\} = \inf\{P(\langle q, \Phi\rangle) - \langle \alpha, q\rangle : q \in \mathbb{R}_+^\kappa\},$$

with the identity also satisfied for $\alpha \in \Delta(\{\partial_q p : q \in \mathbb{R}_+^\kappa\})$;

4. *for each* $\alpha \in \{\partial_q p : q \in \mathbb{R}_+^\kappa\} \cap \operatorname{rel\,int} F(\mathcal{M}_f)$, *we have*

$$\inf\{P(\langle q, \Phi\rangle) - \langle \alpha, q\rangle : q \in \mathbb{R}_+^\kappa\} = \sup\{h_\mu(f) : \mu \in \mathcal{M}_f, F(\mu) = \alpha\}.$$

Proof. By (9.16), we have $B(\Phi_i) \neq -\infty$ for $i = 1, \ldots, \kappa$. Therefore, by Proposition 9.2.1,

$$\max\left\{\lim_{n\to\infty} \frac{1}{n}\int_X \varphi_{i,n}\, d\mu : \mu \in \mathcal{M}_f\right\} = B(\Phi_i) \in \mathbb{R}.$$

Since $h(f) < \infty$, it follows from the variational principle in Theorem 7.2.1 that the function p takes only finite values. For its convexity, we note that given $q, q' \in \mathbb{R}_+^\kappa$ and $t \in (0,1)$, it follows again from Theorem 7.2.1 that

$$\begin{aligned}P(\langle tq + (1-t)q', \Phi\rangle) &= \sup_\mu \big(h_\mu(f) + \langle tq + (1-t)q', F(\mu)\rangle\big) \\ &= \sup_\mu \big(h_\mu(f) + t\langle q, F(\mu)\rangle + (1-t)\langle q', F(\mu)\rangle\big) \\ &\le t \sup_\mu (h_\mu(f) + \langle q, F(\mu)\rangle) + (1-t)\sup_\mu (h_\mu(f) + \langle q', F(\mu)\rangle) \\ &= tP(\langle q, \Phi\rangle) + (1-t)P(\langle q', \Phi\rangle).\end{aligned}$$

Moreover, by Proposition 9.2.1, we have

$$-\infty < B(\langle q, \Phi\rangle) \le p(q) \le h(f) + B(\langle q, \Phi\rangle) < +\infty$$

for every $q \in \mathbb{R}^\kappa_+$. In particular, the function p is locally bounded, and since it is convex it is also continuous.

To establish the second property in the theorem we first obtain some auxiliary results. Given $\alpha \in \mathbb{R}^\kappa$ and $\varepsilon > 0$, we define

$$G(\alpha, n, \varepsilon) = \left\{ x \in X : \left| \frac{\varphi_{i,m}(x)}{m} - \alpha_i \right| < \varepsilon \text{ for } i = 1, \ldots, \kappa \text{ and } m \geq n \right\}.$$

Lemma 9.3.3. *If $G(\alpha, n, \varepsilon) \neq \varnothing$, then for each $q \in \mathbb{R}^\kappa_+$ we have*

$$h(f|G(\alpha, n, \varepsilon)) \leq p(q) - \sum_{i=1}^{\kappa} (\alpha_i - \varepsilon) q_i.$$

Proof of the lemma. Take $s < h(f|G(\alpha, n, \varepsilon))$. By Theorem 4.2.7 (and using the same notation), we obtain

$$\lim_{\delta \to 0} \liminf_{m \to \infty} \frac{1}{m} \log R_m\big(G(\alpha, n, \varepsilon), 0, \delta\big) > s.$$

Hence, given $t < s$, for each sufficiently small $\delta > 0$ there exists $N \in \mathbb{N}$ such that

$$\frac{1}{m} \log R_m\big(G(\alpha, n, \varepsilon), 0, \delta\big) > t$$

for every $m \geq N$. Therefore, for each $m \geq N$, there exists an (m, δ)-separated set $E_m \subset G(\alpha, n, \varepsilon)$ such that $\operatorname{card} E_m > e^{mt}$. Since

$$P(\langle q, \Phi \rangle) = \lim_{\delta \to 0} \limsup_{m \to \infty} \frac{1}{m} \log \sup_F \sum_{x \in E} \exp \sum_{i=1}^{\kappa} q_i \varphi_{i,m}(x),$$

where the supremum is taken over all (m, δ)-separated sets $F \subset X$, we obtain

$$\begin{aligned} P(\langle q, \Phi \rangle) &\geq \limsup_{\delta \to 0} \limsup_{m \to \infty} \frac{1}{m} \log \sum_{x \in E_m} \exp \sum_{i=1}^{\kappa} q_i \varphi_{i,m}(x) \\ &\geq \limsup_{\delta \to 0} \limsup_{m \to \infty} \frac{1}{m} \log \left(e^{mt} \exp \sum_{i=1}^{\kappa} q_i (\alpha_i - \varepsilon) m \right) \\ &= t + \sum_{i=1}^{\kappa} q_i (\alpha_i - \varepsilon) m, \end{aligned}$$

Letting $t \to s$ and $s \to h(f|G(\alpha, n, \varepsilon))$ yields the desired inequality. ☐

By Lemma 9.3.3 and the third property in Theorem 4.2.1, since

$$E_\varepsilon(\alpha) \subset \bigcup_{n=1}^{\infty} G(\alpha, n, \varepsilon),$$

for each $q \in \mathbb{R}_+^\kappa$ we have

$$\begin{aligned} h(f|E_\varepsilon(\alpha)) &\leq h\left(f|\bigcup_{n=1}^\infty G(\alpha, n, \varepsilon)\right) \\ &= \sup_{n\in\mathbb{N}} h(f|G(\alpha, n, \varepsilon)) \\ &\leq p(q) - \sum_{i=1}^\kappa (\alpha_i - \varepsilon) q_i. \end{aligned} \tag{9.17}$$

Lemma 9.3.4. *Given $t \in \mathbb{R}_+^\kappa$ and $\alpha \in \exp(\partial_t p)$, for each $\varepsilon > 0$ there exists an ergodic measure $\mu \in \mathcal{M}_f$ such that*

$$\|F(\mu) - \alpha\| < \varepsilon \quad \textit{and} \quad |h_\mu(f) - (p(t) - \langle \alpha, t\rangle)| < \varepsilon. \tag{9.18}$$

Proof of the lemma. We first assume that p is differentiable at t. Take $\alpha = \nabla p(t)$ and $\delta > 0$. Since p is continuous, there exists $\gamma > 0$ such that

$$\|p(t+s) - p(t) - \langle \alpha, s\rangle\| < \delta\|s\| \tag{9.19}$$

for every $s \in \mathbb{R}^\kappa$ with $\|s\| \leq \gamma$.

Now we observe that by the variational principle in Theorem 7.2.1, there exists $\nu \in \mathcal{M}_f$ such that

$$h_\nu(f) + \langle t, F(\nu)\rangle > p(t) - \delta.$$

It thus follows from Theorem 9.1.1 that if τ is an ergodic decomposition of ν, then

$$\int_{\mathcal{M}_f} \big(h_\mu(f) + \langle t, F(\mu)\rangle\big)\, d\tau(\mu) > p(t) - \delta.$$

Hence, there exists at least one ergodic measure $\mu \in \mathcal{M}_f$ such that

$$h_\mu(f) + \langle t, F(\mu)\rangle > p(t) - \delta. \tag{9.20}$$

On the other hand, also by the variational principle, we have

$$h_\mu(f) + \langle t+s, F(\mu)\rangle \leq p(t+s) \tag{9.21}$$

for every $s \in \mathbb{R}^\kappa$ with $t + s \in \mathbb{R}_+^\kappa$. Therefore, by (9.20) and (9.21),

$$p(t+s) - p(t) \geq \langle s, F(\mu)\rangle - \delta, \tag{9.22}$$

for the same values of s. Now we define points $s_i \in \mathbb{R}^\kappa$ for $i = 1, \ldots, \kappa$ by $s_i = (s_{i1}, \ldots, s_{i\kappa})$, where

$$s_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ \gamma & \text{if } i = j. \end{cases}$$

Taking $s = \pm s_i$ in (9.22) yields

$$\frac{p(t+s_i)-p(t)}{\gamma} \geq \lim_{n\to\infty} \frac{1}{n}\int_X \varphi_{i,n}\,d\mu - \frac{\delta}{\gamma}$$

and

$$\frac{p(t-s_i)-p(t)}{-\gamma} \leq \lim_{n\to\infty} \frac{1}{n}\int_X \varphi_{i,n}\,d\mu + \frac{\delta}{\gamma}.$$

By (9.19), this implies that

$$\left|\lim_{n\to\infty} \frac{1}{n}\int_X \varphi_{i,n}\,d\mu - \alpha_i\right| \leq \delta + \frac{\delta}{\gamma} =: \delta'$$

for $i = 1, \ldots, \kappa$. By (9.20) and (9.21), we thus obtain

$$\begin{aligned} h_\mu(f) &\geq p(t) - \delta - \sum_{i=1}^{\kappa} t_i(\alpha_i + \delta') \\ &\geq p(t) - \langle \alpha, t\rangle - \delta - \kappa\|t\|\delta' \end{aligned}$$

and

$$\begin{aligned} h_\mu(f) &\leq p(t) - \langle t, F(\mu)\rangle \\ &\leq p(t) - \sum_{i=1}^{\kappa} t_i(\alpha_i - \delta') \\ &\leq p(t) - \langle \alpha, t\rangle + \kappa\|t\|\delta'. \end{aligned}$$

This establishes the desired result when p is differentiable at t.

Now we consider the general case, which requires an auxiliary result.

Lemma 9.3.5. *For each $\alpha \in \operatorname{ext}(\partial_t p)$ there is a sequence $(t_n)_{n\in\mathbb{N}} \subset \mathbb{R}^\kappa_+$ converging to t such that $\nabla p(t_n)$ is well defined for each $n \in \mathbb{N}$ and $\nabla p(t_n) \to \alpha$ when $n \to \infty$.*

Proof of the lemma. We recall that a point $x \in C$ is called an *exposed point* of C if there exists a supporting hyperplane H for C such that $H \cap C = \{x\}$. We note that exposed points are extreme points. Let us also recall Straszewicz's theorem (see for example [163]), which says that for any closed convex set $C \subset \mathbb{R}^\kappa$, the set of exposed points of C is dense in the set of extreme points of C.

Due to the former observation, it is sufficient to establish the desired statement when α is an exposed point of $\partial_t p$. So let us take $q \in \mathbb{R}^\kappa$ such that

$$\langle q, \beta\rangle < \langle q, \alpha\rangle \quad \text{for every} \quad \beta \in \partial_t p \setminus \{\alpha\}. \tag{9.23}$$

Since a convex function is differentiable almost everywhere, there exists a sequence $(t_n)_{n\in\mathbb{N}} \subset \mathbb{R}^\kappa_+$ converging to t such that p is differentiable at each t_n, and

$$\left|t_n - \left(t + \frac{q}{n}\right)\right| < \frac{1}{n^2} \quad \text{for all} \quad n \in \mathbb{N}. \tag{9.24}$$

We write $\alpha_n = \nabla p(t_n)$. Since the sequence $(t_n)_{n\in\mathbb{N}}$ is bounded, the same happens with $(\alpha_n)_{n\in\mathbb{N}}$. Hence, without loss of generality, we can assume that $\alpha_n \to \beta$ when $n \to \infty$, for some $\beta \in \mathbb{R}^\kappa$. Since $\alpha_n = \nabla p(t_n)$, we have

$$p(s) - p(t_n) \geq \langle \alpha_n, s - t_n \rangle \quad \text{for all} \quad s \in \mathbb{R}^\kappa_+.$$

Therefore, letting $n \to \infty$ yields

$$p(s) - p(t) \geq \langle \beta, s - t \rangle \quad \text{for all} \quad s \in \mathbb{R}^\kappa_+,$$

which shows that $\beta \in \partial_t p$. On the other hand, we have

$$p(t) - p(t_n) \geq \langle \alpha_t, t - t_n \rangle \quad \text{and} \quad p(t_n) - p(t) \geq \langle \alpha, t_n - t \rangle$$

for every $n \in \mathbb{N}$. Thus,

$$\langle \alpha_n, t_n - t \rangle \geq p(t_n) - p(t) \geq \langle \alpha, t_n - t \rangle.$$

Multiplying this inequality by n and letting $n \to \infty$, it follows from (9.24) that $\langle \alpha', q \rangle \geq \langle \alpha, q \rangle$. Since $\alpha' \in \partial_t p$, it thus follows from (9.23) that $\alpha' = \alpha$. □

By Lemma 9.3.5, given $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that

$$\|\nabla p(t_n) - \alpha\| < \frac{\varepsilon}{2} \quad \text{and} \quad \left|\bigl(p(t_n) - \langle \nabla p(t_n), t_n \rangle\bigr) - \bigl(p(t) - \langle \alpha, t \rangle\bigr)\right| < \frac{\varepsilon}{2}. \quad (9.25)$$

By the former argument for points of differentiability of p, there exists an ergodic measure $\mu \in \mathcal{M}_f$ such that

$$\|F(\mu) - \nabla p(t_n)\| < \frac{\varepsilon}{2} \quad \text{and} \quad \left|h_\mu(f) - (p(t_n) - \langle \nabla p(t_n), t_n \rangle)\right| < \frac{\varepsilon}{2}. \quad (9.26)$$

The desired statement follows now readily from (9.25) and (9.26). □

We proceed with the proof of the second property in the theorem. Take $t \in \mathbb{R}^\kappa_+$, $\alpha \in \operatorname{ext}(\partial_t p)$, and $\varepsilon > 0$. By Lemma 9.3.4, there exists an ergodic measure $\mu \in \mathcal{M}_f$ satisfying (9.18). Since μ is ergodic, it follows from Theorem 9.1.1 that for the functions λ_i in (9.14) we have

$$\lambda_i(x) = \lim_{n\to\infty} \frac{1}{n} \int_X \varphi_{i,n} \, d\mu$$

for μ-almost every $x \in X$ and $i = 1, \ldots, \kappa$. Therefore, $\mu(E(F(\mu))) = 1$, and hence, $\mathcal{E}(F(\mu)) \geq h_\mu(f)$. It thus follows from (9.18) that

$$\begin{aligned} h(f|E_\varepsilon(\alpha)) &\geq h(f|E(F(\mu))) \\ &\geq h_\mu(f) \geq p(t) - \langle \alpha, t \rangle - \varepsilon. \end{aligned}$$

On the other hand, by (9.17) we have

$$h(f|E_\varepsilon(\alpha)) \le p(t) - \sum_{i=1}^{\kappa}(\alpha_i - \varepsilon)t_i.$$

Therefore, letting $\varepsilon \to 0$ yields

$$\begin{aligned}\lim_{\varepsilon\to 0} h(f|E_\varepsilon(\alpha)) &= p(t) - \langle \alpha, t\rangle \\ &\ge \inf\bigl\{p(q) - \langle \alpha, q\rangle : q \in \mathbb{R}_+^\kappa\bigr\}.\end{aligned} \tag{9.27}$$

Since $\alpha \in \partial_t p$, we have $p(t) - \langle\alpha, t\rangle \le p(q) - \langle \alpha, q\rangle$ for every $q \in \mathbb{R}_+^\kappa$, and thus the inequality in (9.27) is in fact an identity.

Now let us take

$$\alpha \in \Delta(\{\mathrm{ext}(\partial_q p) : q \in \mathbb{R}_+^\kappa\}).$$

Then there exist $t_n \in \mathbb{R}_+^\kappa$ and $\beta_n \in \mathrm{ext}(\partial_{t_n} p)$ for each $n \in \mathbb{N}$ such that $\alpha \ge \beta_n$ for every $n \in \mathbb{N}$ and $\beta_n \to \alpha$ when $n \to \infty$. Given $\varepsilon > 0$, we take n sufficiently large so that $\|\alpha - \beta_n\| < \varepsilon/2$. Then

$$\begin{aligned}h(f|E_\varepsilon(\alpha)) &\ge h\bigl(f|\{E(\beta) : \|\beta - \beta_n\| < \varepsilon/2\}\bigr) \\ &\ge p(t_n) - \langle \beta_{t_n}, t_n\rangle \\ &\ge p(t_n) - \langle \alpha, t_n\rangle \\ &\ge \inf\bigl\{p(q) - \langle\alpha, q\rangle : q \in \mathbb{R}_+^\kappa\bigr\}.\end{aligned}$$

Letting $\varepsilon \to 0$ thus yields

$$\lim_{\varepsilon\to 0} h(f|E_\varepsilon(\alpha)) \ge \inf\bigl\{p(q) - \langle\alpha, q\rangle : q \in \mathbb{R}_+^\kappa\bigr\}.$$

On the other hand, by (9.17), we also have

$$\lim_{\varepsilon\to 0} h(f|E_\varepsilon(\alpha)) \le \inf\bigl\{p(q) - \langle\alpha, q\rangle : q \in \mathbb{R}_+^\kappa\bigr\}.$$

This completes the proof of the second property in the theorem.

Now we establish the third property. Take $q \in \mathbb{R}_+^\kappa$ and $\varepsilon > 0$. By Theorem 7.2.1, for each measure $\mu \in \mathcal{M}_f$ with $\|F(\mu) - \alpha\| < \varepsilon$ we have

$$h_\mu(f) \le p(q) - \langle q, F(\mu)\rangle \le p(q) - \sum_{i=1}^{\kappa}(\alpha_i - \varepsilon)q_i.$$

Therefore,

$$c := \lim_{\varepsilon\to 0} \sup\bigl\{h_\mu(f) : \mu \in \mathcal{M}_f, \|F(\mu) - \alpha\| < \varepsilon\bigr\} \le p(q) - \sum_{i=1}^{\kappa} \alpha_i q_i,$$

and hence,

$$c \le \inf\left\{p(q) - \langle \alpha, q\rangle : q \in \mathbb{R}_+^\kappa\right\}.$$

To obtain a lower bound for c, let us take $t \in \mathbb{R}_+^\kappa$ such that $\alpha \in \partial_t p$. We recall Minkowski's theorem, which says that any nonempty compact convex subset C of $\mathbb{R}^\kappa$ is the convex hull of $\operatorname{ext} C$. Then, given $\varepsilon > 0$, there exist $N \in \mathbb{N}$, $\alpha_j \in \operatorname{ext}(\partial_t p)$ and $\lambda_j \in [0,1]$ for $j = 1, \ldots, N$ (in fact, one can take $N = \kappa + 1$) such that

$$\sum_{j=1}^N \lambda_j = 1 \quad \text{and} \quad \alpha = \sum_{j=1}^N \lambda_j \alpha_j.$$

By Lemma 9.3.4, there exist ergodic measures $\mu_j \in \mathcal{M}_f$ for $j = 1, \ldots, N$, such that

$$\|F(\mu_j) - \alpha_j\| < \varepsilon \quad \text{and} \quad \left|h_{\mu_j}(f) - (p(t) - \langle \alpha_j, t\rangle)\right| < \varepsilon \tag{9.28}$$

for each j. For the measure $\mu = \sum_{j=1}^N \mu_j \in \mathcal{M}_f$, we have

$$F(\mu) = \sum_{j=1}^N \lambda_j F(\mu_j) \quad \text{and} \quad h_\mu(f) = \sum_{j=1}^N \lambda_j h_{\mu_j}(f).$$

It thus follows from (9.28) that

$$\|F(\mu) - \alpha\| < \varepsilon \quad \text{and} \quad \left|h_\mu(f) - (p(t) - \langle \alpha, t\rangle)\right| < \varepsilon.$$

Hence,

$$\sup\left\{h_\nu(f) : \nu \in \mathcal{M}_f, \|F(\nu) - \alpha\| < \varepsilon\right\} \ge h_\mu(f) \ge p(t) - \langle \alpha, t\rangle - \varepsilon,$$

and letting $\varepsilon \to 0$ we obtain

$$c \ge p(t) - \langle \alpha, t\rangle \ge \inf\left\{p(q) - \langle \alpha, q\rangle : q \in \mathbb{R}_+^\kappa\right\}.$$

Now let us take $\alpha \in \Delta(\{\partial_q p : q \in \mathbb{R}_+^\kappa\})$. Then there exist $t_n \in \mathbb{R}_+^\kappa$ and $\beta_n \in \operatorname{ext}(\partial_{t_n} p)$ for each $n \in \mathbb{N}$ such that $\alpha \ge \beta_n$ for every $n \in \mathbb{N}$ and $\beta_n \to \alpha$ when $n \to \infty$. Given $\varepsilon > 0$, we take n sufficiently large so that $\|\alpha - \beta_n\| < \varepsilon/2$. Then

$$\begin{aligned}
&\sup\left\{h_\mu(f) : \mu \in \mathcal{M}_f, \|F(\mu) - \alpha\| < \varepsilon\right\} \\
&\ge \sup\left\{h_\mu(f) : \mu \in \mathcal{M}_f, \|F(\mu) - \beta_n\| < \varepsilon/2\right\} \\
&\ge p(t_n) - \langle \beta_n, t_n\rangle \\
&\ge p(t_n) - \langle \alpha, t_n\rangle \\
&\ge \inf\left\{p(q) - \langle \alpha, q\rangle : q \in \mathbb{R}_+^\kappa\right\},
\end{aligned}$$

and letting $\varepsilon \to 0$ yields

$$c \ge \inf\left\{p(q) - \langle \alpha, q\rangle : q \in \mathbb{R}_+^\kappa\right\}.$$

This completes the proof of the third property in the theorem.

For the last property, we consider the function $g\colon F(\mathcal{M}_f) \to \mathbb{R}$ defined by

$$g(\alpha) = \sup\big\{h_\mu(f) : \mu \in \mathcal{M}_f, F(\mu) = \alpha\big\}.$$

We note that $F(\mathcal{M}_f) \subset \mathbb{R}^\kappa$ is a nonempty convex set and that g is concave. Then the function $h\colon \mathbb{R}^\kappa \to \mathbb{R} \cup \{+\infty\}$ defined by

$$h(\alpha) = \begin{cases} -g(\alpha), & \alpha \in F(\mathcal{M}_f), \\ +\infty, & \alpha \notin F(\mathcal{M}_f) \end{cases} \tag{9.29}$$

is convex. Now we consider its Legendre transform

$$\begin{aligned} h^*(q) &= \sup\big\{\langle q, \alpha\rangle - h(\alpha) : \alpha \in \mathbb{R}^\kappa\big\} \\ &= \sup\big\{\langle q, \alpha\rangle + g(\alpha) : \alpha \in F(\mathcal{M}_f)\big\}, \end{aligned}$$

as well as

$$\begin{aligned} h^{**}(q) &= \sup\big\{\langle \alpha, q\rangle - h^*(q) : q \in \mathbb{R}^\kappa\big\} \\ &= -\inf\big\{h^*(q) - \langle \alpha, q\rangle : q \in \mathbb{R}^\kappa\big\}. \end{aligned}$$

We recall that for a convex function $h\colon \mathbb{R}^\kappa \to \mathbb{R}\cup\{+\infty\}$ that is not identically $+\infty$, if h is lower semicontinuous at a point α_0, that is,

$$\liminf_{\alpha\to\alpha_0} h(\alpha) \ge h(\alpha_0),$$

then $h^{**}(\alpha_0) = h(\alpha_0)$ (see for example [163]). Since the function h in (9.29) is lower semicontinuous on the relative interior $\operatorname{rel\,int} F(\mathcal{M}_f)$, we obtain $h^{**}(\alpha) = h(\alpha)$ for every $\alpha \in \operatorname{rel\,int} F(\mathcal{M}_f)$. That is,

$$\inf\big\{h^*(q) - \langle \alpha, q\rangle : q \in \mathbb{R}^\kappa\big\} = g(\alpha)$$

for $\alpha \in \operatorname{rel\,int} F(\mathcal{M}_f)$. On the other hand, by Theorem 7.2.1, we have $p(q) = h^*(q)$ for every $q \in \mathbb{R}^\kappa_+$. Now take $\alpha \in \partial_q p \cap \operatorname{rel\,int} F(\mathcal{M}_f)$. Then $\alpha \in \partial_q h^* \cap \operatorname{rel\,int} F(\mathcal{M}_f)$, and thus,

$$\begin{aligned} g(\alpha) &= \inf\big\{h^*(q') - \langle \alpha, q'\rangle : q' \in \mathbb{R}^\kappa\big\} \\ &= h^*(q) - \langle \alpha, q\rangle \\ &= p(q) - \langle \alpha, q\rangle \\ &= \inf\big\{p(q') - \langle \alpha, q'\rangle : q' \in \mathbb{R}^\kappa_+\big\}. \end{aligned}$$

This completes the proof of the theorem. □

9.4 Multidimensional spectra

We describe in this section a generalization of the multifractal analysis of the entropy spectra in Section 9.2 to multidimensional sequences, that is, to vectors of asymptotically subadditive sequences.

In order to formulate the theorem, given $\delta > 0$ we define

$$\Delta_\delta = \Delta\big(\{\partial_t p : t_i \geq \delta \text{ for } i = 1, \ldots, \kappa\}\big),$$

with $\Delta(A)$ as in Section 9.3, and with the function p as in (9.15).

Theorem 9.4.1 ([66]). *Let $f\colon X \to X$ be a continuous transformation of a compact metric space with $h(f) < \infty$ such that the entropy map $\mu \mapsto h_\mu(f)$ is upper semicontinuous. If $\Phi_1, \ldots, \Phi_\kappa$ are asymptotically subadditive sequences satisfying* (9.16)*, then:*

1. *for each $t \in \mathbb{R}^\kappa_+$, if $\alpha \in \operatorname{ext}(\partial_t p)$ then $E(\alpha) \neq \varnothing$, and*

$$\mathcal{E}(\alpha) = \inf\big\{P\big(\langle q, \Phi\rangle\big) - \langle \alpha, q\rangle : q \in \mathbb{R}^\kappa_+\big\} = P\big(\langle t, \Phi\rangle\big) - \langle \alpha, t\rangle; \qquad (9.30)$$

2. *for each $\alpha \in \bigcup_\delta \Delta_\delta$ we have*

$$\inf\big\{P(\langle q, \Phi\rangle) - \langle \alpha, q\rangle : q \in \mathbb{R}^\kappa_+\big\} = \max\big\{h_\mu(f) : \mu \in \mathcal{M}_f,\ F(\mu) = \alpha\big\};$$

3. *if for some $t \in \mathbb{R}^\kappa_+$ the sequence $\langle t, \Phi\rangle$ has a unique equilibrium measure μ_t, then μ_t is ergodic,*

$$\nabla p(t) = F(\mu_t), \quad E(\nabla p(t)) \neq \varnothing, \quad \textit{and} \quad \mathcal{E}(\nabla p(t)) = h_{\mu_t}(f).$$

Proof. We first formulate a multidimensional version of Theorem 7.4.2.

Lemma 9.4.2. *The following properties hold:*

1. *for each $q \in \mathbb{R}^\kappa_+$, the set $C_{\langle q,\Phi\rangle} \subset \mathcal{M}_f$ of all equilibrium measures for $\langle q, \Phi\rangle$ is a nonempty compact convex set;*
2. *each extreme point of $C_{\langle q,\Phi\rangle}$ is an ergodic measure;*
3. *for each $q \in \mathbb{R}^\kappa_+$ we have*

$$\partial_q p = \big\{F(\mu) : \mu \in C_{\langle q,\Phi\rangle}\big\}.$$

Proof of the lemma. The first two statements follow readily from Theorem 7.4.2. The last statement can be obtained by repeating arguments in the proof of the third statement of Theorem 7.4.2, with the function f in (7.24) replaced by the function $f\colon \mathbb{R}^\kappa_+ \times \mathcal{M}_f \to \mathbb{R}$ defined by

$$f(q, \mu) = h_\mu(f) + \langle q, F(\mu)\rangle.$$

Then the sets $I(q)$ in (7.25) and

$$R(q) = \{F(\mu) : \mu \in C_{\langle q,\Phi\rangle}\}$$

are nonempty convex subsets respectively of $\mathcal{M}_f$ and $\mathbb{R}^\kappa$, for each $q \in \mathbb{R}^\kappa_+$. Only a slight change is required in the last part of the proof of Theorem 7.4.2, when it is shown that $\partial_q p \subset R(q)$ (incidentally, we note that when $\kappa = 1$ the set $\partial_p f$ coincides with the interval $[p'(q^-), p'(q^+)]$). Namely, and using the same notation, the number γ' is now chosen as follows. Since $B_\delta(R(q))$ is a compact convex set (because the same happens with $R(q)$), there exists $v \in \mathbb{R}^\kappa$ with $\|v\| = 1$ such that $\langle a, v\rangle > \langle b, v\rangle$ for every $b \in B_\delta(R(q))$. On the other hand, by a multidimensional version of Lemma 7.4.4, there exists $\gamma > 0$ such that $R(t) \subset B_\delta(R(q))$ whenever $\|t - q\| < \gamma$. Now we take $\gamma' \in (0, \gamma/2)$ such that

$$t_0 := q + \gamma' v \in \mathbb{R}^\kappa_+.$$

Then $\langle a, t_0 - q\rangle > \langle b, t_0 - q\rangle$ for $b \in R(t_0)$, and again we obtain a contradiction. □

We proceed with the proof of the theorem. Take $t \in \mathbb{R}^\kappa_+$. By Lemma 9.4.2, the set $C_{\langle t,\Phi\rangle}$ of all equilibrium measures for $\langle t, \Phi\rangle$ is a nonempty compact convex set. Now we observe that for each $\alpha \in \mathbb{R}^\kappa$, the set

$$D(t, \alpha) = \{\mu \in C_{\langle t,\Phi\rangle} : F(\mu) = \alpha\}$$

is compact and convex. The convexity follows readily from the linearity of the map $\mu \mapsto F(\mu)$. For the compactness, let us take a sequence $(\mu_n)_{n\in\mathbb{N}} \subset D(t, \alpha)$ converging to some measure $\mu \in \mathcal{M}_f$. By the upper semicontinuity of the maps $\mu \mapsto h_\mu(f)$ and

$$\mu \mapsto \lim_{n\to\infty} \frac{1}{n} \int_X \varphi_{i,n}\, d\mu \tag{9.31}$$

for $i = 1, \ldots, \kappa$, we obtain

$$h_\mu(f) + \langle t, F(\mu)\rangle \geq \limsup_{n\to\infty} \big(h_{\mu_n}(f) + \langle t, F(\mu_n)\rangle\big) = p(t).$$

It follows from the variational principle in Theorem 7.2.1 that the measure μ is an equilibrium measure for $\langle t, \Phi\rangle$, that is,

$$h_\mu(f) + \langle t, F(\mu)\rangle = p(t). \tag{9.32}$$

Moreover, again by the upper semicontinuity of the functions above, we have

$$\begin{aligned} h_\mu(f) &\geq \limsup_{n\to\infty} h_{\mu_n}(f) \\ &= \lim_{n\to\infty} \big(p(t) - \langle t, F(\mu_n)\rangle\big) \\ &= p(t) - \langle t, \alpha\rangle, \end{aligned} \tag{9.33}$$

and

$$\lim_{m\to\infty}\frac{1}{m}\int_X \varphi_{i,m}\,d\mu \geq \limsup_{n\to\infty}\lim_{m\to\infty}\frac{1}{m}\int_X \varphi_{i,m}\,d\mu_n \\ = \limsup_{n\to\infty}\alpha_i = \alpha_i \tag{9.34}$$

for $i = 1, \dots, \kappa$. Moreover, since $t \in \mathbb{R}_+^\kappa$, if the inequality in (9.34) is strict for some i, then

$$\langle t, F(\mu)\rangle = \sum_{i=1}^{\kappa} t_i \lim_{m\to\infty}\frac{1}{m}\int_X \varphi_{i,m}\,d\mu \\ > \sum_{i=1}^{\kappa} t_i\alpha_i = \langle t, \alpha\rangle.$$

Together with (9.33) this yields

$$h_\mu(f) + \langle t, F(\mu)\rangle > p(t),$$

which contradicts (9.32). Therefore, we have an identity in (9.34) for each $i = 1, \dots, \kappa$, that is, $F(\mu) = \alpha$. Thus, we have shown that $\mu \in D(t, \alpha)$, and hence, the set $D(t, \alpha)$ is compact.

Now let us take $\alpha \in \operatorname{ext}(\partial_t p)$. By the third property in Lemma 9.4.2, the set $D(t, \alpha)$ is nonempty. We recall the Krein–Milman theorem, which says that any nonempty compact convex subset C of a locally convex Hausdorff topological vector space is the closed convex hull of $\operatorname{ext} C$. Since $D(t, \alpha)$ is a nonempty compact convex set, it contains at least one extreme point, say μ. If $\mu = a\mu_1 + (1-a)\mu_2$ for some $\mu_1, \mu_2 \in \mathcal{M}_f$ and $a \in (0, 1)$, then

$$p(t) = h_\mu(f) + \langle t, F(\mu)\rangle \\ = a\big(h_{\mu_1}(f) + \langle t, F(\mu_1)\rangle\big) + (1-a)\big(h_{\mu_2}(f) + \langle t, F(\mu_2)\rangle\big).$$

It thus follows from the variational principle in Theorem 7.2.1 that the measures μ_1 and μ_2 are equilibrium measures for $\langle t, \Phi\rangle$. Therefore, by Lemma 9.4.2, we have $F(\mu_1), F(\mu_2) \in \partial_t p$. Since $\alpha \in \operatorname{ext}(\partial_t p)$ and

$$\alpha = F(\mu) = aF(\mu_1) + (1-a)F(\mu_2),$$

we obtain $F(\mu_1) = F(\mu_2) = \alpha$. In other words, $\mu_1, \mu_2 \in D(t, \alpha)$. Since μ is an extreme point of $D(\alpha, t)$, we conclude that $\mu_1 = \mu_2 = \mu$. That is, μ is an extreme point of $\mathcal{M}_f$, and hence it is ergodic. By the second property in Theorem 9.1.1, we have $\mu(E(\alpha)) = 1$, and thus,

$$h(f|E(\alpha)) \geq h_\mu(f) = p(t) - \langle t, \alpha\rangle \\ = \inf\big\{p(q) - \langle q, \alpha\rangle : q \in \mathbb{R}_+^\kappa\big\}. \tag{9.35}$$

On the other hand, it follows from (9.17) that

$$h(f|E(\alpha)) \le h(f|E_\varepsilon(\alpha)) \le p(q) - \sum_{i=1}^{\kappa}(\alpha_i - \varepsilon)q_i,$$

and letting $\varepsilon \to 0$ we obtain

$$h(f|E(\alpha)) \le p(q) - \langle q, \alpha\rangle.$$

Together with (9.35) this establishes identity (9.30) in the theorem.

Now we establish the second property. Take $\alpha \in \bigcup_\delta \Delta_\delta$. In view of the third property in Theorem 9.3.2, it is sufficient to show that

$$\inf\left\{p(q) - \langle\alpha, q\rangle : q \in \mathbb{R}_+^\kappa\right\} \le \max\left\{h_\mu(f) : \mu \in \mathcal{M}_f, F(\mu) = \alpha\right\}. \tag{9.36}$$

We first assume that $\alpha \in \partial_t p$ for some $t \in \mathbb{R}_+^\kappa$. Again by the third property in Theorem 9.3.2, there exists a sequence $(\mu_n)_{n\in\mathbb{N}} \subset \mathcal{M}_f$ such that

$$\lim_{n\to\infty} F(\mu_n) = \alpha \quad \text{and} \quad \limsup_{n\to\infty} h_{\mu_n}(f) \ge \inf\left\{p(q) - \langle\alpha, q\rangle : q \in \mathbb{R}_+^\kappa\right\}.$$

Without loss of generality, we may assume that the sequence $(\mu_n)_{n\in\mathbb{N}}$ converges to some measure $\mu \in \mathcal{M}_f$. By the upper semicontinuity of the entropy and of the functions in (9.31), we obtain

$$\begin{aligned} h_\mu(f) &\ge \limsup_{n\to\infty} h_{\mu_n}(f) \\ &\ge \inf\left\{p(q) - \langle q, \alpha\rangle : q \in \mathbb{R}_+^\kappa\right\} \\ &= p(t) - \langle t, \alpha\rangle, \end{aligned}$$

and

$$F(\mu) \ge \limsup_{n\to\infty} F(\mu_n) = \alpha.$$

Hence, by Theorem 7.2.1,

$$\begin{aligned} h_\mu(f) &\ge p(t) - \langle t, \alpha\rangle \\ &\ge p(t) - \langle t, F(\mu)\rangle \ge h_\mu(f). \end{aligned}$$

This implies that $F(\mu) = \alpha$ and $h_\mu(f) = p(t) - \langle t, \alpha\rangle$.

For an arbitrary $\alpha \in \bigcup_\delta \Delta_\delta$, there exists a sequence $(t_n)_{n\in\mathbb{N}} \subset \mathbb{R}_+^\kappa$ such that each component of t_n is greater than δ, for every $n \in \mathbb{N}$, and there exists $\alpha_n \in \operatorname{ext}(\partial_{t_n} p)$ for each $n \in \mathbb{N}$ such that $\alpha \ge \alpha_n$ for every $n \in \mathbb{N}$ and $\alpha_n \to \alpha$ when $n \to \infty$. By the former paragraph, for each $n \in \mathbb{N}$ there exists $\mu_n \in \mathcal{M}_f$ such that

$$F(\mu_n) = \alpha_n \quad \text{and} \quad h_{\mu_n}(f) = p(t_n) - \langle\alpha_n, t_n\rangle.$$

Without loss of generality, we may also assume that the sequence $(\mu_n)_{n\in\mathbb{N}}$ converges to some measure $\mu \in \mathcal{M}_f$. Then

$$F(\mu) \geq \limsup_{n\to\infty} F(\mu_n) = \alpha, \tag{9.37}$$

and

$$\begin{aligned} h_\mu(f) &\geq \limsup_{n\to\infty} h_{\mu_n}(f) \\ &= \limsup_{n\to\infty} \bigl(p(t_n) - \langle \alpha_n, t_n\rangle\bigr) \\ &\geq \limsup_{n\to\infty} \bigl(p(t_n) - \langle \alpha, t_n\rangle\bigr) \\ &\geq \limsup_{n\to\infty} \bigl(h_\mu(f) + \langle F(\mu) - \alpha, t_n\rangle\bigr) \\ &\geq h_\mu(f) + \sum_{i=1}^{\kappa} \left(\lim_{m\to\infty} \frac{1}{m} \int_X \varphi_{i,m}\, d\mu - \alpha_i \right) \delta, \end{aligned}$$

using Theorem 7.2.1 in the third inequality. Since $\delta > 0$, this implies that

$$\lim_{m\to\infty} \frac{1}{m} \int_X \varphi_{i,m}\, d\mu \leq \alpha_i$$

for $i = 1, \ldots, \kappa$. It thus follows from (9.37) that $F(\mu) = \alpha$. Moreover,

$$\begin{aligned} h_\mu(f) &\geq \limsup_{n\to\infty} \bigl(p(t_n) - \langle \alpha, t_n\rangle\bigr) \\ &\geq \inf\bigl\{p(q) - \langle \alpha, q\rangle : q \in \mathbb{R}_+^\kappa\bigr\}. \end{aligned}$$

This establishes inequality (9.36), and thus also the second property in the theorem.

For the last property, let us take $t \in \mathbb{R}_+^\kappa$ such that $\langle t, \Phi\rangle$ has a unique equilibrium measure μ_t. By Lemma 9.4.2, we have $\partial_t p = \{F(\mu_t)\}$. That is, p is differentiable at t, with $\nabla p(t) = F(\mu_t)$. We have also shown above that each set $D(t, \alpha)$ contains at least one ergodic measure, and hence μ_t ergodic. Moreover, it follows from the former properties in the theorem that $E(\nabla p(t)) \neq \varnothing$ and

$$\mathcal{E}(\nabla p(t)) = p(t) - \langle \alpha, t\rangle = h_{\mu_t}(f).$$

This completes the proof of the theorem. ∎

Part IV

Almost Additive Thermodynamic Formalism

Chapter 10

Almost Additive Sequences

We consider in this chapter the particular class of almost additive sequences and we develop to a larger extent the nonadditive thermodynamic formalism in this setting. We note that any almost additive sequence is asymptotically subadditive, in the sense of Chapter 7, and thus we may expect stronger thermodynamic properties than those in the general nonadditive thermodynamic formalism as well as in the subadditive thermodynamic formalism. This includes a discussion of the existence and uniqueness of equilibrium and Gibbs measures, both for repellers and for hyperbolic sets. On the other hand, the class of almost additive sequences is still sufficiently general to allow nontrivial applications, in particular to the multifractal analysis of the Lyapunov exponents associated to nonconformal repellers (see Chapter 11). Further applications to multifractal analysis are described in Chapter 12. In order to avoid unnecessary technicalities, we first develop the theory for repellers. We then explain how the proofs of the corresponding results for hyperbolic sets and more generally for continuous maps with upper semicontinuous entropy can be obtained from the proofs for repellers. In particular, we describe some regularity properties of the topological pressure for continuous maps with upper semicontinuous entropy.

10.1 Repellers

As we already mentioned above, we first develop the theory for repellers. The main advantage is that we avoid some of the technicalities that are needed in order to obtain corresponding results for more general classes of maps. Moreover, the results presented in this section can then be used directly in the study of nonconformal repellers in Chapter 11.

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10.1.1 Pressure for almost additive sequences

We introduce in this section the notion of almost additive sequence. We also give alternative formulas for the nonadditive topological pressure of an almost additive sequence, in particular in terms of the periodic points.

We first introduce the class of almost additive sequences. Let $f\colon J \to J$ be a continuous map of a compact metric space.

Definition 10.1.1. A sequence of functions $\Phi = (\varphi_n)_{n\in\mathbb{N}}$ is said to be *almost additive* (with respect to f in J) if there is a constant $C > 0$ such that

$$-C + \varphi_n(x) + \varphi_m(f^n(x)) \le \varphi_{n+m}(x) \le C + \varphi_n(x) + \varphi_m(f^n(x))$$

for every $n, m \in \mathbb{N}$ and $x \in J$.

Clearly, any additive sequence of functions $\varphi_n = \sum_{k=0}^{n-1} \varphi \circ f^k$ is almost additive. Some nontrivial examples of almost additive sequences, related to the study of Lyapunov exponents of nonconformal transformations, are given in Section 11.2. We note that in view of Proposition 7.1.6, any almost additive sequence is asymptotically additive.

Now let J be a repeller of a C^1 map f. Given a Markov partition $R_1, \ldots, R_\kappa$ of J, we consider the associated topological Markov chain $\sigma|\Sigma_A^+$ with transition matrix $A = (a_{ij})$ given by (5.2), with h replaced by f. We also consider the sets

$$R_{i_1\cdots i_n} = \bigcap_{k=0}^{n-1} f^{-k} R_{i_{k+1}}. \tag{10.1}$$

For each $n \in \mathbb{N}$, we let

$$\gamma_n(\Phi) = \sup\left\{|\varphi_n(x) - \varphi_n(y)| : x, y \in R_{i_1\cdots i_n} \text{ and } (i_1 \cdots i_n) \in S_n\right\}, \tag{10.2}$$

where S_n is the set of all Σ_A^+-admissible sequences of length n. We note that $\gamma_n(\Phi)$ coincides with the number $\gamma_n(\Phi, \mathcal{U})$ in (4.1) for the cover $\mathcal{U}$ of J formed by the elements of the Markov partition. We have the following equivalence.

Proposition 10.1.2. *For an almost additive sequence Φ, the following properties are equivalent:*

1. *Φ has tempered variation;*
2. *$\gamma_n(\Phi)/n \to 0$ when $n \to \infty$.*

Proof. By the remark after (10.2), it is clear that if the second property holds, then Φ has tempered variation. In the other direction, let us assume that Φ has tempered variation. Let also $\mathcal{U}_l$ be the cover of J formed by the sets $R_{i_1\cdots i_l}$. We note that

$$\begin{aligned}
|\varphi_{n+l-1}(x) - \varphi_{n+l-1}(y)| \le{}& |\varphi_{n+l-1}(x) - \varphi_n(x) - \varphi_{l-1}(f^n(x))| \\
&+ |\varphi_n(x) + \varphi_{l-1}(f^n(x)) - \varphi_n(y) - \varphi_{l-1}(f^n(x))| \\
&+ |\varphi_n(y) + \varphi_{l-1}(f^n(x)) - \varphi_{n+l-1}(y)| \\
\le{}& 2C + |\varphi_n(x) - \varphi_n(y)| + 2\|\varphi_{l-1}\|_\infty,
\end{aligned}$$

with the convention that $\varphi_0 = 0$. Therefore,

$$\gamma_{n+l-1}(\Phi) = \gamma_{n+l-1}(\Phi, \mathcal{U}_1) \le 2C + \gamma_n(\Phi, \mathcal{U}_l) + 2\|\varphi_{l-1}\|_\infty,$$

and hence,

$$\begin{aligned}\limsup_{n\to\infty} \frac{\gamma_n(\Phi)}{n} &= \limsup_{n\to\infty} \frac{\gamma_{n+l-1}(\Phi, \mathcal{U}_1)}{n} \\ &\le \limsup_{n\to\infty} \frac{\gamma_n(\Phi, \mathcal{U}_l)}{n}.\end{aligned}$$

Since Φ has tempered variation, letting $l \to \infty$ we obtain

$$\limsup_{n\to\infty} \frac{\gamma_n(\Phi)}{n} \le \lim_{l\to\infty} \limsup_{n\to\infty} \frac{\gamma_n(\Phi, \mathcal{U}_l)}{n} = 0.$$

This completes the proof of the proposition. □

The following result of Barreira and Gelfert in [10] gives a formula for the nonadditive topological pressure of an almost additive sequence.

Theorem 10.1.3. *Let J be a repeller of a C^1 map, and let $\Phi = (\varphi_n)_{n\in\mathbb{N}}$ be an almost additive sequence of continuous functions in J with tempered variation. Then*

$$P_J(\Phi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{i_1\cdots i_n} \exp \varphi_n(x_{i_1\cdots i_n}) \tag{10.3}$$

for any points $x_{i_1\cdots i_n} \in R_{i_1\cdots i_n}$, for each $(i_1 \cdots i_n) \in S_n$ and $n \in \mathbb{N}$.

Proof. Since $\Phi = (\varphi_n)_{n\in\mathbb{N}}$ is almost additive, the sequence $\Psi = (\psi_n)_{n\in\mathbb{N}}$ defined by $\psi_n = \varphi_n + C$ is subadditive. Now we consider the cover $\mathcal{U}$ of J formed by the rectangles $R_1, \ldots, R_\kappa$ of the Markov partition used to define the sets $R_{i_1\cdots i_n}$. By (4.13), we have

$$\mathcal{Z}_n(J, \Phi, \mathcal{U}) = \sum_{i_1\cdots i_n} \exp \max_{x\in R_{i_1\cdots i_n}} \varphi_n(x),$$

where the sum is taken over all sequences $(i_1 \cdots i_n) \in S_n$. Since the sequence Φ has tempered variation, the same happens with Ψ. It thus follows from Theorem 4.2.6 that

$$P_J(\Psi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{i_1\cdots i_n} \exp \max_{x\in R_{i_1\cdots i_n}} \psi_n(x). \tag{10.4}$$

Since Ψ has tempered variation, there exists a sequence $(\rho_n)_{n\in\mathbb{N}} \subset \mathbb{R}^+$ decreasing to zero such that

$$\psi_n(x_{i_1\cdots i_n}) \ge \max_{x\in R_{i_1\cdots i_n}} \psi(x) - n\rho_n$$

for any points $x_{i_1\cdots i_n} \in R_{i_1\cdots i_n}$, for each $(i_1 \cdots i_n) \in S_n$ and $n \in \mathbb{N}$. Therefore,

$$\frac{1}{n} \log \sum_{i_1\cdots i_n} \exp \psi_n(x_{i_1\cdots i_n}) \geq -\rho_n + \frac{1}{n} \log \sum_{i_1\cdots i_n} \exp \max_{x \in R_{i_1\cdots i_n}} \psi_n(x),$$

and hence, by (10.4),

$$\liminf_{n\to\infty} \frac{1}{n} \log \sum_{i_1\cdots i_n} \exp \psi_n(x_{i_1\cdots i_n}) \geq P_J(\Psi).$$

It also follows from (10.4) that

$$\limsup_{n\to\infty} \frac{1}{n} \log \sum_{i_1\cdots i_n} \exp \psi_n(x_{i_1\cdots i_n}) \leq P_J(\Psi).$$

Combining the two inequalities we conclude that

$$\begin{aligned} P_J(\Psi) &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{i_1\cdots i_n} \exp \psi_n(x_{i_1\cdots i_n}) \\ &= C + \lim_{n\to\infty} \frac{1}{n} \log \sum_{i_1\cdots i_n} \exp \varphi_n(x_{i_1\cdots i_n}), \end{aligned}$$

and since $P_J(\Psi) = P_J(\Phi) + C$ we obtain identity (10.3). □

We note that identity (10.3) ensures not only that for an almost additive sequence the nonadditive topological pressure is a limit, but also that this limit is independent of the particular Markov partition used to compute it.

We also describe an alternative characterization of the topological pressure which has the advantage of not requiring the symbolic dynamics. Let

$$\mathrm{Fix}(f) = \{x \in J : f(x) = x\}$$

be the set of fixed points of f in J.

Theorem 10.1.4 ([6]). *Let J be a repeller of a C^1 map, and let $\Phi = (\varphi_n)_{n\in\mathbb{N}}$ be an almost additive sequence of continuous functions in J with tempered variation. Then*

$$P_J(\Phi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{x \in \mathrm{Fix}(f^n)} \exp \varphi_n(x).$$

Proof. Since

$$\varphi_n \geq \varphi_{n-k} + \varphi_k \circ f^{n-k} - C \geq \varphi_{n-k} + \sum_{j=0}^{k-1} \varphi_1 \circ f^{n-k+j} - kC,$$

we have

$$\sum_{x\in\mathrm{Fix}(f^n)} e^{\varphi_n(x)} \ge \sum_{i_1\cdots i_n} a_{i_n i_1} e^{\varphi_n(x_{i_1\cdots i_n})-\gamma_n(\Phi)}$$
$$\ge \sum_{i_1\cdots i_{n-k}} e^{\varphi_{n-k}(x_{i_1\cdots i_{n-k}})-\gamma_n(\Phi)-\gamma_{n-k}(\Phi)-k(\|\varphi_1\|_\infty+C)},$$

where a_{ij} are the entries of the $\kappa\times\kappa$ transition matrix associated to the Markov partition. Hence, by (10.3) and the tempered variation of Φ, we obtain

$$\liminf_{n\to\infty}\frac{1}{n}\log\sum_{x\in\mathrm{Fix}(f^n)}\exp\varphi_n(x)\ge P_J(\Phi).$$

Furthermore,

$$\sum_{x\in\mathrm{Fix}(f^n)} e^{\varphi_n(x)} \le \sum_{i_1\cdots i_n} a_{i_n i_1} e^{\varphi_n(x_{i_1\cdots i_n})+\gamma_n(\Phi)} \le \sum_{i_1\cdots i_n} e^{\varphi_n(x_{i_1\cdots i_n})+\gamma_n(\Phi)}.$$

Again by (10.3) and the tempered variation of Φ, we obtain

$$\limsup_{n\to\infty}\frac{1}{n}\log\sum_{x\in\mathrm{Fix}(f^n)}\exp\varphi_n(x)\le P_J(\Phi).$$

This completes the proof of the theorem. □

10.1.2 Variational principle

Here and in the following sections we develop the thermodynamic formalism for almost additive sequences, always for repellers. We first establish a variational principle for the topological pressure.

Let again J be a repeller of a C^1 map f. We continue to denote by $\mathcal{M}_f$ the family of f-invariant probability measures in J, and by $h_\mu(f)$ the entropy of f with respect to a measure $\mu\in\mathcal{M}_f$. We note that in the present situation the entropy can be computed as follows. For the partition

$$\xi_n=\big\{R_{i_1\cdots i_n}:(i_1\cdots i_n)\in S_n\big\}$$

of the repeller J, we set

$$H_\mu(\xi_n)=-\sum_{i_1\cdots i_n}\mu(R_{i_1\cdots i_n})\log\mu(R_{i_1\cdots i_n}).$$

Then

$$h_\mu(f)=\lim_{n\to\infty}\frac{1}{n}H_\mu(\xi_n)=\inf_{n\in\mathbb{N}}\frac{1}{n}H_\mu(\xi_n). \tag{10.5}$$

The following is a variational principle for the nonadditive topological pressure of an almost additive sequence. Although it was first obtained by Barreira in [6], it is essentially a consequence of the variational principle for the topological pressure of an asymptotically subadditive sequence in Theorem 7.2.1.

Theorem 10.1.5. *Let J be a repeller of a C^1 map f, and let $\Phi = (\varphi_n)_{n\in\mathbb{N}}$ be an almost additive sequence of continuous functions in J with tempered variation. Then*

$$\begin{aligned} P_J(\Phi) &= \max_{\mu\in\mathcal{M}_f}\left(h_\mu(f) + \int_J \lim_{n\to\infty}\frac{\varphi_n(x)}{n}\,d\mu(x)\right) \\ &= \max_{\mu\in\mathcal{M}_f}\left(h_\mu(f) + \lim_{n\to\infty}\frac{1}{n}\int_J \varphi_n\,d\mu\right). \end{aligned} \tag{10.6}$$

Proof. We provide two proofs. One is based on the application of Theorem 7.2.1. The other one follows closely the original arguments in [6].

First proof. In view of Proposition 7.1.6, it follows readily from Theorem 7.2.1 that

$$P_J(\Phi) = \sup_{\mu\in\mathcal{M}_f}\left(h_\mu(f) + \lim_{n\to\infty}\frac{1}{n}\int_J \varphi_n\,d\mu\right). \tag{10.7}$$

On the other hand, since the sequence $\varphi_n + C$ is subadditive, it follows from Kingman's subadditive ergodic theorem that for each measure $\mu \in \mathcal{M}_f$ the limit

$$\psi(x) := \lim_{n\to\infty}\frac{\varphi_n(x)}{n}$$

exists for μ-almost every $x \in J$. Furthermore,

$$-Cn + \sum_{k=0}^{n-1}\varphi_1\circ f^k \le \varphi_n \le Cn + \sum_{k=0}^{n-1}\varphi_1\circ f^k.$$

Therefore, $|\varphi_n/n| \le C + \|\varphi_1\|_\infty$, and by Lebesgue's dominated convergence theorem, we obtain

$$\lim_{n\to\infty}\frac{1}{n}\int_J\varphi_n\,d\mu = \int_J\psi\,d\mu = \int_J\lim_{n\to\infty}\frac{\varphi_n(x)}{x}\,d\mu(x).$$

Together with (10.7) this yields

$$P_J(\Phi) = \sup_{\mu\in\mathcal{M}_f}\left(h_\mu(f) + \int_J \lim_{n\to\infty}\frac{\varphi_n(x)}{n}\,d\mu(x)\right). \tag{10.8}$$

Now we show that the suprema in (10.7) and (10.8) are always attained. We first observe that since the sequence of functions $\varphi_n + C$ is subadditive, the same happens with the sequence $\int_J(\varphi_n + C)\,d\mu$, and thus,

$$\begin{aligned} \lim_{n\to\infty}\frac{1}{n}\int_J\varphi_n\,d\mu &= \lim_{n\to\infty}\frac{1}{n}\int_J(\varphi_n + C)\,d\mu \\ &\le \frac{1}{n}\int_J(\varphi_n + C)\,d\mu = \frac{1}{n}\int_J\varphi_n\,d\mu + \frac{C}{n}. \end{aligned}$$

Similarly, the sequence $\int_J (\varphi_n - C)\,d\mu$ is supadditive, and we obtain

$$\lim_{n\to\infty} \frac{1}{n}\int_J \varphi_n\,d\mu \geq \frac{1}{n}\int_J \varphi_n\,d\mu - \frac{C}{n}.$$

Therefore,

$$\left|\lim_{n\to\infty} \frac{1}{n}\int_J \varphi_n\,d\mu - \frac{1}{n}\int_J \varphi_n\,d\mu\right| \leq \frac{C}{n}. \tag{10.9}$$

If $(\mu_m)_{m\in\mathbb{N}}$ is a sequence of measures converging to μ, then

$$\left|\lim_{n\to\infty} \frac{1}{n}\int_J \varphi_n\,d\mu_m - \frac{1}{n}\int_J \varphi_n\,d\mu_m\right| \leq \frac{C}{n}$$

for every $m, n \in \mathbb{N}$. Letting $m \to \infty$ and then $n \to \infty$ we obtain

$$\lim_{m\to\infty}\lim_{n\to\infty} \frac{1}{n}\int_J \varphi_n\,d\mu_m = \lim_{n\to\infty} \frac{1}{n}\int_J \varphi_n\,d\mu. \tag{10.10}$$

The desired statement follows now readily from the continuity in (10.10) and from the upper semicontinuity of the entropy map $\mu \mapsto h_\mu(f)$.

Second proof. Now we follow the original arguments in [6], using the estimate of Feng and Huang in (7.3) for asymptotically additive sequences (we recall that by Proposition 7.1.6, any almost additive sequence is asymptotically additive).

Given numbers $c_1, \ldots, c_k$ and $p_1, \ldots, p_k \geq 0$ satisfying $\sum_{i=1}^k p_i = 1$, we have

$$0 \geq \sum_{i=1}^k p_i \left(c_i - \log p_i - \log \sum_{i=1}^k e^{c_i}\right)$$

(see Lemma 2.3.3). Now we set

$$c_{i_1\cdots i_n} = \max\{\exp \varphi_n(y) : y \in R_{i_1\cdots i_n}\},$$

with the convention that $c_{i_1\cdots i_n} = 0$ if $R_{i_1\cdots i_n} = \varnothing$, and

$$\alpha_n = \sum_{i_1\cdots i_n} c_{i_1\cdots i_n}. \tag{10.11}$$

For each measure $\mu \in \mathcal{M}_f$ we have

$$\begin{aligned} 0 &\geq \sum_{i_1\cdots i_n} \mu(R_{i_1\cdots i_n})(\log c_{i_1\cdots i_n} - \log \mu(R_{i_1\cdots i_n}) - \log \alpha_n) \\ &= H_\mu(\xi_n) + \sum_{i_1\cdots i_n} \mu(R_{i_1\cdots i_n})(\log c_{i_1\cdots i_n} - \log \alpha_n), \end{aligned}$$

and hence,

$$\log \alpha_n \geq H_\mu(\xi_n) + \int_J \varphi_n \, d\mu.$$

Dividing by n, and taking the limit when $n \to \infty$, it follows from (10.5), (10.11), and Theorem 10.1.3 that

$$P_J(\Phi) \geq h_\mu(f) + \lim_{n\to\infty} \frac{1}{n} \int_J \varphi_n \, d\mu.$$

Therefore,

$$P_J(\Phi) \geq \sup_{\mu \in \mathcal{M}_f} \left(h_\mu(f) + \lim_{n\to\infty} \frac{1}{n} \int_J \varphi_n \, d\mu \right).$$

Now we establish the reverse inequality. Given $x \in J$ and $n \in \mathbb{N}$, we define a probability measure in J by

$$\mu_{x,n} = (\delta_x + \delta_{f(x)} + \cdots + \delta_{f^{n-1}(x)})/n,$$

where δ_y is the probability delta-measure at y. Let also $V(x)$ be the set of all sublimits of the sequence $(\mu_{x,n})_{n\in\mathbb{N}}$. Clearly, $V(x) \subset \mathcal{M}_f$ and $V(x) \neq \varnothing$ for every $x \in J$ (by the Krylov–Bogoliubov theorem).

Let us take $x \in J$ and $\mu \in V(x)$. Let also $(m_n)_{n\in\mathbb{N}}$ be an increasing sequence such that the sequence of measures $(\mu_{x,m_n})_{n\in\mathbb{N}}$ converges to μ. By (7.3), for each $\varepsilon > 0$ there exist $K > 0$ and $C_{\varepsilon,k} > 0$ for each $k \geq K$ such that

$$\left| \varphi_n(x) - \frac{1}{k} \sum_{j=0}^{n-1} \varphi_k(f^j(x)) \right| \leq n\varepsilon + C_{\varepsilon,k}$$

for every $n \geq 2k$. This implies that

$$\left| \frac{\varphi_n(x)}{n} - \frac{1}{k} \int_J \varphi_k \, d\mu_{x,n} \right| = \left| \frac{\varphi_n(x)}{n} - \frac{1}{kn} \sum_{j=0}^{n-1} \varphi_k(f^j(x)) \right| \leq \varepsilon + \frac{C_{\varepsilon,k}}{n}. \tag{10.12}$$

Replacing n by m_n in (10.12) and letting $n \to \infty$ yields

$$\begin{aligned} \frac{1}{k} \int_J \varphi_k \, d\mu - \varepsilon &\leq \liminf_{n\to\infty} \frac{\varphi_{m_n}(x)}{m_n} \\ &\leq \limsup_{n\to\infty} \frac{\varphi_{m_n}(x)}{m_n} \leq \frac{1}{k} \int_J \varphi_k \, d\mu + \varepsilon. \end{aligned} \tag{10.13}$$

Therefore, setting

$$\gamma := \lim_{k\to\infty} \frac{1}{k} \int_J \varphi_k \, d\mu,$$

and letting $k \to \infty$ in (10.13) we obtain

$$\gamma - \varepsilon \leq \liminf_{n\to\infty} \frac{\varphi_{m_n}(x)}{m_n} \leq \limsup_{n\to\infty} \frac{\varphi_{m_n}(x)}{m_n} \leq \gamma + \varepsilon.$$

Finally, it follows from the arbitrariness of ε that

$$\lim_{n\to\infty}\frac{\varphi_{m_n}(x)}{m_n}=\gamma. \tag{10.14}$$

In particular, given $\delta>0$, we can always assume that

$$\left|\frac{\varphi_{m_n}(x)}{m_n}-\gamma\right|\le\delta \quad\text{for}\quad n\in\mathbb{N}. \tag{10.15}$$

Now we proceed as in the proof of Theorem 4.3.1. Given a finite set E, we consider again the quantities $\mu_a(e)$ and $H(a)$ in (4.28) and (4.29). Using (10.15), one can show that there exist $m,n\in\mathbb{N}$ with n arbitrarily large and a set $R_{i_1\cdots i_N}$ containing x such that:

1. $(i_1\cdots i_N)$ contains a subvector $(j_1\cdots j_{km})$ with $km\ge N-m$, for which

   $$H(j_1\cdots j_{km})\le m(h_\mu(f)+\delta)$$

   with respect to the set

   $$E=\{R_{k_1\cdots k_m}:(k_1\cdots k_m)\in S_m\};$$

2. we have

   $$\sup_{y\in R_{i_1\cdots i_N}}\varphi_N(y)-\gamma_N(\Phi)\le\varphi_N(x)\le N(\gamma+\delta).$$

Take $\delta>0$. Given $m\in\mathbb{N}$ and $u\in\mathbb{R}$, we denote by $J_{m,u}$ the set of points $x\in J$ satisfying properties 1 and 2 for some measure $\mu\in V(x)$ with $\gamma\in[u-\delta,u+\delta]$, that is, with

$$u-\delta\le\lim_{n\to\infty}\frac{1}{n}\int_J\varphi_n\,d\mu\le u+\delta.$$

We also denote by b_N the number of vectors of length N satisfying properties 1 and 2 for some point $x\in J_{m,u}$. By property 1, we have

$$b_N\le p^m\operatorname{card}\{(j_1\cdots j_{km}):H(j_1\cdots j_{km})\le m(h_\mu(f)+\delta)\}.$$

In a similar manner to that in the proof of Lemma 4.3.3, one can show that

$$b_N\le\exp\left[N(h_\mu(f)+2\delta)\right]$$

for all sufficiently large N. We notice that for each $l\in\mathbb{N}$, the vectors $(i_1\cdots i_N)$ satisfying properties 1 and 2 for some $x\in J_{m,u}$ and some $N\ge l$ cover the set $J_{m,u}$.

For the cover $\mathcal{U}=\{R_1,\ldots,R_\kappa\}$ obtained from the Markov partition, we have

$$P_Z(\Phi)=\inf\{\alpha\in\mathbb{R}:M_Z(\alpha,\Phi,\mathcal{U})=0\}.$$

Moreover, it follows from the former discussion that

$$\begin{aligned}M_{J_{m,u}}(\alpha,\Phi,\mathcal{U}) &\le \limsup_{l\to\infty}\sum_{N=l}^{\infty} b_N \exp\left[-\alpha N + N(\gamma+\delta)+\gamma_N(\Phi)\right]\\ &\le \limsup_{l\to\infty}\sum_{N=l}^{\infty}\exp\left[N\left(h_\mu(f)+\gamma+4\delta-\alpha+\limsup_{n\to\infty}\frac{\gamma_n(\Phi)}{n}\right)\right].\end{aligned}$$

Setting

$$c=\sup_{\mu\in\mathcal{M}_f}\left(h_\mu(f)+\lim_{n\to\infty}\frac{1}{n}\int_J \varphi_n\,d\mu\right),$$

since the sequence Φ has tempered variation we obtain

$$M_{J_{m,u}}(\alpha,\Phi,\mathcal{U}) \le \limsup_{l\to\infty}\sum_{N=l}^{\infty}\exp[N(c+4\delta-\alpha)].$$

Therefore, $M_{J_{m,u}}(\alpha,\Phi,\mathcal{U})=0$ whenever $\alpha>c+4\delta$, and hence $P_{J_{m,u}}(\Phi)\le c+4\delta$.

Let us take points $u_1,\ldots,u_r$ such that for each number u with $|u|\le\gamma$ there exists $j\in\{1,\ldots,r\}$ with $|u-u_j|<\delta$. Then

$$J=\bigcup_{m\in\mathbb{N}}\bigcup_{i=1}^{r} J_{m,u_i},$$

and it follows from Theorem 4.2.1 that

$$P_J(\Phi)=\sup_{m,i} P_{J_{m,u_i}}(\Phi)\le c+4\delta.$$

Since δ is arbitrary, we conclude that

$$P_J(\Phi)\le \sup_{\mu\in\mathcal{M}_f}\left(h_\mu(f)+\lim_{n\to\infty}\frac{1}{n}\int_J\varphi_n\,d\mu\right).$$

This establishes the identities in (10.6) with each maximum replaced by a supremum. The remaining argument to show that each supremum is indeed a maximum is identical to that in the first proof. □

The variational principle in Theorem 10.1.5 was first established by Barreira and Gelfert in [10] in the particular case of almost additive sequences with bounded variation (see Definition 10.1.8).

10.1.3 Equilibrium and Gibbs measures

We consider in this section the notions of equilibrium measure and of Gibbs measure for an almost additive sequence. In particular, we study the existence and uniqueness of these measures, and we present several characterizations of the uniqueness of equilibrium measures (see also Section 10.1.4).

According to Definition 7.4.1, a measure $\mu \in \mathcal{M}_f$ is said to be an *equilibrium measure* for the almost additive sequence Φ (with respect to f in J) if it attains any of the maxima in (10.6) (and thus both maxima), that is, if

$$P_J(\Phi) = h_\mu(f) + \lim_{n\to\infty} \frac{1}{n} \int_J \varphi_n \, d\mu.$$

The existence of equilibrium measures is already contained in Theorem 10.1.5 (notice that there are maxima in (10.6), not suprema).

Theorem 10.1.6. *Let J be a repeller of a C^1 map. Then any almost additive sequence of continuous functions in J with tempered variation has at least one equilibrium measure.*

Now we introduce the notion of Gibbs measure.

Definition 10.1.7. A probability measure μ in J is said to be a *Gibbs measure* for the sequence Φ (with respect to f in J) if there exists a constant $K > 0$ such that

$$K^{-1} \le \frac{\mu(R_{i_1\cdots i_n})}{\exp[-nP_J(\Phi) + \varphi_n(x)]} \le K$$

for every $n \in \mathbb{N}$, $(i_1 \cdots i_n) \in S_n$, and $x \in R_{i_1\cdots i_n}$.

We note that a Gibbs measure need not be invariant. On the other hand, one can easily verify that, as in the classical theory, any invariant Gibbs measure is also an equilibrium measure. For this, we first note that if μ is an f-invariant Gibbs measure, then

$$h_\mu(x) := \lim_{n\to\infty} -\frac{1}{n} \log \mu(R_{i_1\cdots i_n}) = P_J(\Phi) - \lim_{n\to\infty} \frac{\varphi_n(x)}{n}$$

for μ-almost every $x \in J$. By the Shannon–McMillan–Breiman theorem, we obtain

$$h_\mu(f) = \int_J h_\mu(x)\, d\mu(x) = P_J(\Phi) - \int_J \lim_{n\to\infty} \frac{\varphi_n(x)}{n}\, d\mu(x),$$

and hence μ is an equilibrium measure.

For the following result we need the stronger notion of bounded variation.

Definition 10.1.8. A sequence of functions $\Phi = (\varphi_n)_{n\in\mathbb{N}}$ is said to have *bounded variation* if $\sup_{n\in\mathbb{N}} \gamma_n(\Phi) < \infty$, with $\gamma_n(\Phi)$ as in (10.2).

For example, if Φ is the additive sequence $\varphi_n = \sum_{k=0}^{n-1} \varphi \circ f^k$, for some Hölder continuous function φ in a repeller, then one can easily verify that Φ has bounded variation. Of course, bounded variation implies tempered variation.

Given a sequence of continuous functions $\Phi = (\varphi_n)_{n\in\mathbb{N}}$ in J with bounded variation, we set

$$c_{i_1\cdots i_n} = \max\{\exp\varphi_n(y) : y \in R_{i_1\cdots i_n}\},$$

with the convention that $c_{i_1\cdots i_n} = 0$ if $R_{i_1\cdots i_n} = \varnothing$. We also set

$$\alpha_n = \sum_{i_1\cdots i_n} c_{i_1\cdots i_n}.$$

We then define a probability measure ν_n in the algebra generated by the sets $R_{i_1\cdots i_n}$ by

$$\nu_n(R_{i_1\cdots i_n}) = c_{i_1\cdots i_n}/\alpha_n \tag{10.16}$$

for each $(i_1\cdots i_n) \in S_n$, and we extend it arbitrarily to the Borel σ-algebra of J (we continue to denote the extension by ν_n). Let also $\mathcal{M}(\Phi)$ be the set of all sublimits of the sequence of measures $(\nu_n)_{n\in\mathbb{N}}$ (in the Borel σ-algebra). Clearly, $\mathcal{M}(\Phi) \neq \varnothing$.

The following result shows that for an almost additive sequence with bounded variation there exists a unique equilibrium measure, and that this measure is a Gibbs measure.

Theorem 10.1.9 ([6]). *Let J be a repeller of a C^1 map which is topologically mixing on J, and let Φ be an almost additive sequence of continuous functions in J with bounded variation. Then:*

1. *there is a unique equilibrium measure for Φ;*
2. *there is a unique invariant Gibbs measure for Φ;*
3. *the two measures coincide and are ergodic.*

Proof. We follow closely [6], although some arguments are borrowed from [10]. Since the map f is topologically mixing on J, there exists $q \in \mathbb{N}$ such that A^q has only positive entries, where A is the transition matrix associated to the Markov partition. Setting $k = q - 1$, this ensures that given sequences $(i_1\cdots i_n) \in S_n$ and $(j_1\cdots j_l) \in S_l$ there exists $(p_1\cdots p_k) \in S_k$ such that

$$(i_1\cdots i_n p_1\cdots p_k j_1\cdots j_l) \in S_{n+k+l}.$$

Lemma 10.1.10. *Each measure $\nu \in \mathcal{M}(\Phi)$ is a Gibbs measure.*

Proof of the lemma. The argument is an elaboration of the proof of Theorem 3.4.4. Take $n \in \mathbb{N}$ and $l > n$. Since the sequence Φ is almost additive, we have

$$c_{i_1\cdots i_n j_1\cdots j_{l-n}} \leq e^C c_{i_1\cdots i_n} c_{j_1\cdots j_{l-n}},$$

and thus,

$$\sum_{j_1\cdots j_{l-n}} c_{i_1\cdots i_n j_1\cdots j_{l-n}} \leq e^C c_{i_1\cdots i_n}\alpha_{l-n}. \tag{10.17}$$

In particular,

$$\alpha_l \le e^C \alpha_n \alpha_{l-n}. \tag{10.18}$$

On the other hand, for each $(j_1 \cdots j_{l-k}) \in S_{l-k}$ there exists $(m_1 \cdots m_k) \in S_k$ such that

$$(i_1 \cdots i_n m_1 \cdots m_k j_1 \cdots j_{l-k}) \in S_{n+l}.$$

Hence, for each $x \in R_{i_1 \cdots i_n m_1 \cdots m_k j_1 \cdots j_{l-k}}$ we have

$$c_{i_1 \cdots i_n k_1 \cdots k_k j_1 \cdots j_{l-k}} \ge e^{-2C} e^{\varphi_n(x)} e^{\varphi_k(f^n(x))} e^{\varphi_{l-k}(f^{n+k}(x))}. \tag{10.19}$$

Now we assume that the point x also satisfies the identity $e^{\varphi_{l-k}(f^{n+k}(x))} = c_{j_1 \cdots j_{l-k}}$. Since Φ has bounded variation, it follows from (10.19) that

$$\begin{aligned} c_{i_1 \cdots i_n m_1 \cdots m_k j_1 \cdots j_{l-k}} &\ge e^{-2C} C_1' e^{\varphi_n(x)} e^{\varphi_{l-k}(f^{n+k}(x))} \\ &\ge e^{-2C} C_1 c_{i_1 \cdots i_n} c_{j_1 \cdots j_{l-k}}, \end{aligned} \tag{10.20}$$

for some universal constants $C_1', C_1 > 0$. Moreover,

$$\begin{aligned} \alpha_l = \sum_{j_1 \cdots j_l} c_{j_1 \cdots j_l} &\le \kappa^k \sum_{j_1 \cdots j_{l-k}} c_{j_1 \cdots j_l} \\ &\le \kappa^k e^{k(C + \|\varphi_1\|_\infty)} \sum_{j_1 \cdots j_{l-k}} c_{j_1 \cdots j_{l-k}} = \left(\kappa e^{C + \|\varphi\|_\infty}\right)^k \alpha_{l-k}. \end{aligned}$$

Thus, it follows from (10.20) that

$$\begin{aligned} \sum_{t_1 \cdots t_l} c_{i_1 \cdots i_n t_1 \cdots t_l} &\ge \sum_{j_1 \cdots j_{l-k}} c_{i_1 \cdots i_n m_1 \cdots m_k j_1 \cdots j_{l-k}} \\ &\ge e^{-2C} C_1 c_{i_1 \cdots i_n} \alpha_{l-k} \ge C_2 c_{i_1 \cdots i_n} \alpha_l, \end{aligned} \tag{10.21}$$

where $C_2 > 0$ is again some universal constant. In particular,

$$\alpha_{n+l} \ge C_2 \alpha_n \alpha_l. \tag{10.22}$$

By (10.18), the sequence $e^C \alpha_n$ is submultiplicative, and hence,

$$\lim_{n\to\infty} \frac{1}{n} \log \alpha_n = \lim_{n\to\infty} \frac{1}{n} \log(e^C \alpha_n) = \inf_{n\in\mathbb{N}} \frac{1}{n} \log(e^C \alpha_n). \tag{10.23}$$

Now we recall that (see (10.3))

$$P_J(\Phi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{i_1 \cdots i_n} c_{i_1 \cdots i_n} = \lim_{n\to\infty} \frac{1}{n} \log \alpha_n.$$

By (10.23), we obtain $\exp[nP_J(\Phi)] \le e^C \alpha_n$. Similarly, by (10.22),

$$P_J(\Phi) = \sup_{n\in\mathbb{N}} \frac{1}{n} \log(C_2 \alpha_n),$$

and thus $\exp[nP_J(\Phi)] \ge C_2\alpha_n$. Therefore, for each $n \in \mathbb{N}$ we have

$$C_2\alpha_n \le \exp[nP_J(\Phi)] \le e^C\alpha_n. \tag{10.24}$$

Since ν_l is a measure in the algebra generated by the sets $R_{i_1\cdots i_l}$, we have

$$\nu_l(R_{i_1\cdots i_n}) = \sum_{j_1\cdots j_{l-n}} \frac{c_{i_1\cdots i_n j_1\cdots j_{l-n}}}{\alpha_l}.$$

By (10.17), (10.22), and (10.24), and using the almost additivity and the bounded variation of Φ, we obtain

$$\begin{aligned}\nu_l(R_{i_1\cdots i_n}) &\le c_{i_1\cdots i_n}\alpha_{l-n}\alpha_l^{-1}e^C \le c_{i_1\cdots i_n}\alpha_n^{-1}e^C C_2\\ &\le c_{i_1\cdots i_n}\exp[-nP_J(\Phi)]C_3\\ &\le \exp[-nP_J(\Phi)+\varphi_n(y)]C_4\end{aligned} \tag{10.25}$$

for every $y \in R_{i_1\cdots i_n}$, where $C_3, C_4 > 0$ are universal constants. Analogously, by (10.18), (10.21), and (10.24), we obtain

$$\begin{aligned}\nu_l(R_{i_1\cdots i_n}) &\ge c_{i_1\cdots i_n}\alpha_{l-n}\alpha_l^{-1}C_2 \ge c_{i_1\cdots i_n}\alpha_n^{-1}C_5\\ &\ge c_{i_1\cdots i_n}\exp[-nP_J(\Phi)]C_5\\ &\ge \exp[-nP_J(\Phi)+\varphi_n(y)]C_6\end{aligned} \tag{10.26}$$

for every $y \in R_{i_1\cdots i_n}$, where $C_5, C_6 > 0$ are again universal constants. Now we consider a sequence $(\nu_{n_k})_{k\in\mathbb{N}}$ converging to a measure ν when $k \to \infty$. Replacing l by n_k in (10.25) and (10.26), and letting $k \to \infty$ we obtain

$$C_6 \le \frac{\nu(R_{i_1\cdots i_n})}{\exp[-nP_J(\Phi)+\varphi_n(y)]} \le C_4$$

for every $y \in R_{i_1\cdots i_n}$. Therefore, ν is a Gibbs measure. □

Lemma 10.1.11. *Any Gibbs measure for Φ is ergodic.*

Proof of the lemma. Let ν be a Gibbs measure for Φ. For each two sets $R_{i_1\cdots i_n}$ and $R_{j_1\cdots j_l}$, given each $m > n + 2k$ we have

$$\nu(R_{i_1\cdots i_n} \cap f^{-m}R_{j_1\cdots j_l}) = \sum_{k_1\cdots k_{m-n}} \nu(R_{i_1\cdots i_n k_1\cdots k_{m-n} j_1\cdots j_l}).$$

Using similar arguments to those in the proof of Lemma 10.1.10, we obtain

$$\begin{aligned}\nu(R_{i_1\cdots i_n} \cap f^{-m}R_{j_1\cdots j_l}) &\ge \sum_{k_1\cdots k_{m-n}} c_{i_1\cdots i_n k_1\cdots k_{m-n} j_1\cdots j_l}\exp[-(m+l)P_J(\Phi)]C_7\\ &\ge \exp[-(m+l)P_J(\Phi)]c_{i_1\cdots i_n}c_{j_1\cdots j_l}\alpha_{m-n}C_8\\ &\ge \exp[-(n+l)P_J(\Phi)]c_{i_1\cdots i_n}c_{j_1\cdots j_l}C_9\\ &\ge \nu(R_{i_1\cdots i_n})\nu(R_{j_1\cdots j_l})C_{10},\end{aligned}$$

where $C_7, C_8, C_9, C_{10} > 0$ are universal constants. Since each Borel set can be arbitrarily approximated in measure by a disjoint union of sets $R_{i_1\cdots i_n}$, possibly not all with the same n, standard arguments of measure theory show that

$$\liminf_{m\to\infty} \nu(A \cap f^{-m}B) \geq C_{10}\nu(A)\nu(B) \tag{10.27}$$

for any Borel sets $A, B \subset J$. Now let A be an f-invariant set and take $B = J \setminus A$. By (10.27), either $\nu(A) = 0$ or $\nu(A) = 1$, and thus, the measure ν is ergodic. $\square$

Now let ν be a Gibbs measure for Φ (by Lemma 10.1.10 it always exists), and let us consider the sequence of measures $\frac{1}{n}\sum_{l=0}^{n-1} \nu \circ f^{-l}$. Since f is continuous, any limit point μ of this sequence is an f-invariant probability measure in J. Furthermore, using similar arguments to those in the proof of Lemma 10.1.10, we obtain

$$\begin{aligned}
\nu(f^{-l}R_{i_1\cdots i_n}) &= \sum_{j_1\cdots j_l} \nu(R_{j_1\cdots j_l i_1\cdots i_n}) \\
&\leq c_1 \sum_{j_1\cdots j_l} \exp[-(l+n)P_J(\Phi)]c_{j_1\cdots j_l i_1\cdots i_n} \\
&\leq c_2 \sum_{j_1\cdots j_l} \exp[-(l+n)P_J(\Phi)]c_{j_1\cdots j_l} c_{i_1\cdots i_n} \\
&= c_2 \exp[-(l+n)P_J(\Phi)]\alpha_l c_{i_1\cdots i_n} \\
&\leq c_3 \exp[-nP_J(\Phi)]c_{i_1\cdots i_n} \leq c_4\nu(R_{i_1\cdots i_n}),
\end{aligned}$$

for some universal constants c_1, c_2, c_3, and $c_4 > 0$. Analogously,

$$\begin{aligned}
\nu(f^{-l}R_{i_1\cdots i_n}) &\geq c_5 \sum_{j_1\cdots j_l} \exp[-(l+n)P_J(\Phi)]c_{j_1\cdots j_l i_1\cdots i_n} \\
&\geq c_6 \sum_{j_1\cdots j_l} \exp[-(l+n)P_J(\Phi)]c_{j_1\cdots j_l} c_{i_1\cdots i_n} \\
&\geq c_7 \exp[-nP_J(\Phi)]c_{i_1\cdots i_n} \geq c_8\nu(R_{i_1\cdots i_n}),
\end{aligned}$$

for some universal constants c_5, c_6, c_7, and $c_8 > 0$. Therefore,

$$c_8\nu(R_{i_1\cdots i_n}) \leq \frac{1}{n}\sum_{l=0}^{n-1} \nu(f^{-l}R_{i_1\cdots i_n}) \leq c_4\nu(R_{i_1\cdots i_n}),$$

and hence also

$$c_8\nu(R_{i_1\cdots i_n}) \leq \mu(R_{i_1\cdots i_n}) \leq c_4\nu(R_{i_1\cdots i_n}), \tag{10.28}$$

for every $n \in \mathbb{N}$ and $(i_1 \cdots i_n) \in S_n$. It follows from (10.28) that μ is a Gibbs measure for Φ. By Lemma 10.1.11, the measure ν is ergodic, and thus, by (10.28), the measure μ is also ergodic. The uniqueness of μ follows from its ergodicity together with the fact that by the Gibbs property any two such measures must be equivalent.

Now let $\nu \in \mathcal{M}_f$ be an equilibrium measure for Φ, and let μ be the unique invariant Gibbs measure for Φ. We want to show that $\nu = \mu$. We first observe that $\nu = \alpha\eta + (1-\alpha)\mu'$ for some $\alpha \in [0,1]$ and some $\eta, \mu' \in \mathcal{M}_f$ such that $\mu' \ll \mu$ and $\eta \perp \mu$. The Radon–Nikodym derivative $d\mu'/d\mu$ is f-invariant μ-almost everywhere. Moreover, by Lemma 10.1.11, the measure μ is ergodic, and hence $d\mu'/d\mu$ is constant μ-almost everywhere. Therefore, $\mu' = \mu$. Since $\eta \perp \mu$, we have

$$h_\nu(f) = \alpha h_\eta(f) + (1-\alpha)h_\mu(f)$$

(see for example Corollary 4.3.17 in [108]). Since

$$\begin{aligned} P_J(\Phi) &= h_\nu(f) + \lim_{n\to\infty} \frac{1}{n} \int_J \varphi_n \, d\nu \\ &= \alpha \left(h_\eta(f) + \lim_{n\to\infty} \frac{1}{n} \int_J \varphi_n \, d\eta \right) \\ &\quad + (1-\alpha) \left(h_\mu(f) + \lim_{n\to\infty} \frac{1}{n} \int_J \varphi_n \, d\mu \right), \end{aligned} \tag{10.29}$$

and μ is an equilibrium measure, either $\alpha = 0$, in which case $\nu = \mu' = \mu$, or, by (10.29), the measure η must also be an equilibrium measure for Φ. We will show that this is impossible, in view of the assumption $\eta \perp \mu$.

If $B \subset J$ is an f-invariant set such that $\mu(B) = 0$ and $\eta(B) = 1$, then given $\delta > 0$ there exists $n = n(\delta) \in \mathbb{N}$ and a union C_n of sets of the form $R_{i_1\cdots i_n}$ such that

$$\mu(B\Delta C_n) < \delta \quad \text{and} \quad \eta(B\Delta C_n) < \delta. \tag{10.30}$$

Since $h_\eta(f) \le H_\eta(\xi_n)/n$, using (10.9) we obtain

$$\begin{aligned} &n \left(h_\eta(f) + \lim_{n\to\infty} \frac{1}{n} \int_J \varphi_n \, d\eta \right) \\ &\le H_\eta(\xi_n) + \int_J \varphi_n \, d\eta + C \\ &= \sum_{i_1\cdots i_n} \left(-\eta(R_{i_1\cdots i_n}) \log \eta(R_{i_1\cdots i_n}) + \int_{R_{i_1\cdots i_n}} \varphi_n \, d\eta \right) + C \\ &\le \gamma_n(\Phi) + C + \sum_{i_1\cdots i_n} \eta(R_{i_1\cdots i_n})[\varphi_n(x_{i_1\cdots i_n}) - \log \eta(R_{i_1\cdots i_n})] \\ &\le \gamma_n(\Phi) + C + \eta(C_n) \log \sum_{R_{i_1\cdots i_n}\cap C_n \neq \varnothing} \exp \varphi_n(x_{i_1\cdots i_n}) \\ &\quad + \eta(J \setminus C_n) \log \sum_{R_{i_1\cdots i_n}\cap C_n = \varnothing} \exp \varphi_n(x_{i_1\cdots i_n}) + \frac{2}{e}, \end{aligned}$$

for any points $x_{i_1\cdots i_n} \in R_{i_1\cdots i_n}$, where we have used the inequality

$$\sum_{i=1}^{k} x_i(b_i - \log x_i) \leq \sum_{i=1}^{k} x_i \log \sum_{j=1}^{k} e^{b_j} + \frac{1}{e},$$

which is valid for any numbers $b_i \in \mathbb{R}$ and $x_i \geq 0$ for $i = 1, \ldots, k$. Therefore, since μ is a Gibbs measure we obtain

$$\begin{aligned} &n\left(h_\eta(f) + \lim_{n\to\infty}\frac{1}{n}\int_J \varphi_n\,d\eta - P_J(\Phi)\right) \\ &\leq \gamma_n(\Phi) + C + \frac{2}{e} \\ &\quad + \eta(C_n)\log \sum_{R_{i_1\cdots i_n}\cap C_n\neq\varnothing} \exp[\varphi_n(x_{i_1\cdots i_n}) - nP_J(\Phi)] \\ &\quad + \eta(J\setminus C_n)\log \sum_{R_{i_1\cdots i_n}\cap C_n=\varnothing} \exp[\varphi_n(x_{i_1\cdots i_n}) - nP_J(\Phi)] \\ &\leq \gamma_n(\Phi) + C + \frac{2}{e} + \eta(C_n)\log[K\mu(C_n)] + \eta(J\setminus C_n)\log[K\mu(J\setminus C_n)], \end{aligned} \tag{10.31}$$

where $K > 0$ is some universal constant. Letting $\delta \to 0$ in (10.30), we obtain $\mu(C_n) \to 0$ and $\eta(C_n) \to 1$ when $n \to \infty$. Therefore, the right-hand side of (10.31) converges to $-\infty$ when $n \to \infty$ (we recall that Φ has bounded variation). This shows that

$$h_\eta(f) + \lim_{n\to\infty}\frac{1}{n}\int_J \varphi_n\,d\eta < P_J(\Phi),$$

and η is not an equilibrium measure for Φ. In particular, this shows that there exists a unique equilibrium measure. Moreover, it coincides with the unique equilibrium measure for Φ (since $\nu = \mu$). This completes the proof of the theorem. □

The next result follows readily from Lemmas 10.1.10 and 10.1.11.

Theorem 10.1.12. *Let J be a repeller of a C^1 map which is topologically mixing on J, and let Φ be an almost additive sequence of continuous functions in J with bounded variation. Then each element of $\mathcal{M}(\Phi)$ is an ergodic Gibbs measure for Φ.*

We note that a priori the set $\mathcal{M}(\Phi)\cap\mathcal{M}_f$ could be empty. It follows from Theorem 10.1.12 and the definition of Gibbs measure that any two measures in $\mathcal{M}(\Phi)$ must be equivalent. Since any two ergodic measures are mutually singular and $\mathcal{M}(\Phi) \neq \varnothing$, it follows from Theorem 10.1.12 that there is at most one *invariant* measure in $\mathcal{M}(\Phi)$. In view of Theorem 10.1.9, we thus obtain the following characterization of the unique invariant Gibbs measure.

Theorem 10.1.13. *Let J be a repeller of a C^1 map which is topologically mixing on J, and let Φ be an almost additive sequence of continuous functions in J with bounded variation. Then the unique invariant Gibbs measure for Φ is the unique invariant measure in $\mathcal{M}(\Phi)$.*

In particular, there exists a unique invariant accumulation point of the sequence of measures $(\nu_n)_{n\in\mathbb{N}}$ in (10.16). Moreover, since the unique equilibrium measure is a Gibbs measure, it follows from the proof of Lemma 10.1.11 that

$$\liminf_{m\to\infty} \mu(A \cap f^{-m} f) \geq C_{10}\mu(A)\mu(B) \tag{10.32}$$

for any Borel sets $A, B \subset J$, where $C_{10} > 0$ is some universal constant. Proceeding in a similar manner to that in the proof of Lemma 10.1.11 one can also show that

$$\limsup_{m\to\infty} \mu(A \cap f^{-m} B) \leq C_{11}\mu(A)\mu(B) \tag{10.33}$$

for any Borel sets $A, B \subset J$, where $C_{11} > 0$ is again some universal constant. Using (10.32) and (10.33), we can proceed in a similar manner to that for example in the proof of Proposition 20.3.6 in [108] to show that the measure μ is mixing.

When Φ is an almost additive sequence of continuous functions in J with tempered variation (and not necessarily with bounded variation), it follows from Theorem 11.5.1 that there exist an ergodic probability measure ν in J, a sequence $(\rho_n)_{n\in\mathbb{N}} \subset \mathbb{R}^+$ decreasing to 0, and $K > 0$ such that

$$(Ke^{n\rho_n})^{-1} \leq \frac{\nu(R_{i_1\cdots i_n})}{\exp[-nP_J(\Phi) + \varphi_n(x)]} \leq Ke^{n\rho_n}$$

for every $n \in \mathbb{N}$, $(i_1 \cdots i_n) \in S_n$, and $x \in R_{i_1\cdots i_n}$. We emphasize that the measure ν need not be invariant.

10.1.4 Equilibrium measures and periodic points

The unique measure in Theorem 10.1.9 can also be characterized in terms of the periodic points, in a similar manner to that in the classical theory. Let δ_x be the delta-measure at x.

Theorem 10.1.14 ([6]). *Let J be a repeller of a C^1 map which is topologically mixing on J, and let $\Phi = (\varphi_n)_{n\in\mathbb{N}}$ be an almost additive sequence of continuous functions in J with bounded variation. Then the unique equilibrium measure for Φ is the limit of the sequence of invariant probability measures*

$$\mu_n = \sum_{x\in\mathrm{Fix}(f^n)} e^{\varphi_n(x)}\delta_x \Big/ \sum_{x\in\mathrm{Fix}(f^n)} e^{\varphi_n(x)}. \tag{10.34}$$

Proof. Each measure ν_n in (10.16), when restricted to the algebra generated by the sets $R_{i_1\cdots i_n}$, can be written in the form

$$\nu_n = \frac{1}{\alpha_n} \sum_{i_1\cdots i_n} c_{i_1\cdots i_n} \chi_{R_{i_1\cdots i_n}},$$

where χ_B denotes the characteristic function of the set B. Let $A = (a_{ij})$ be the transition matrix associated to the Markov partition. We consider an extension $\bar{\nu}_n$ of the measure ν_n to the Borel σ-algebra of J such that

$$\bar{\nu}_n | R_{i_1 \cdots i_n} = \delta_{x_{i_1 \cdots i_n}} / \alpha_n$$

whenever $a_{i_n i_1} = 1$, where $x_{i_1 \cdots i_n} \in \operatorname{Fix}(f^n)$ is the periodic point corresponding to the repetition of the sequence $(i_1 \cdots i_n)$ in the symbolic dynamics. Hence,

$$\bar{\nu}_n \geq \frac{1}{\alpha_n} \sum_{i_1 \cdots i_n} a_{i_n i_1} c_{i_1 \cdots i_n} \delta_{x_{i_1 \cdots i_n}} \geq \frac{1}{\alpha_n} \sum_{x \in \operatorname{Fix}(f^n)} e^{\varphi_n(x)} \delta_x, \tag{10.35}$$

which is the same as $\bar{\nu}_n \geq \beta_n \mu_n / \alpha_n$, where (see (10.34))

$$\mu_n = \frac{1}{\beta_n} \sum_{x \in \operatorname{Fix}(f^n)} e^{\varphi_n(x)} \delta_x \quad \text{and} \quad \beta_n = \sum_{x \in \operatorname{Fix}(f^n)} e^{\varphi_n(x)}.$$

Since

$$\varphi_n \geq \varphi_{n-k} + \varphi_k \circ f^{n-k} - C \geq \varphi_{n-k} + \sum_{j=0}^{k-1} \varphi_1 \circ f^{n-k+j} - kC,$$

setting

$$\gamma = \sup_{n \in \mathbb{N}} \gamma_n(\Phi) + k(\|\varphi_1\|_\infty + C)$$

we obtain

$$\begin{aligned} \beta_n &= \sum_{i_1 \cdots i_n} a_{i_n i_1} e^{\varphi_n(x_{i_1 \cdots i_n})} \\ &\geq \sum_{i_1 \cdots i_{n-k}} c_{i_1 \cdots i_{n-k}} e^{-\gamma_{n-k}(\Phi) - k(\|\varphi_1\|_\infty + C)} \geq \alpha_{n-k} e^{-\gamma}. \end{aligned} \tag{10.36}$$

It follows from (10.18) that

$$\bar{\nu}_n \geq \frac{\alpha_{n-k}}{\alpha_n} e^{-\gamma} \mu_n \geq \frac{e^{-\gamma - C}}{\alpha_k} \mu_n. \tag{10.37}$$

We also establish an upper bound for $\bar{\nu}_n$. Proceeding in a similar manner to that in (10.35), we obtain

$$\bar{\nu}_n \leq \frac{1}{\alpha_n} \sum_{i_1 \cdots i_{n+k}} a_{i_{n+k} i_1} c_{i_1 \cdots i_n} \delta_{x_{i_1 \cdots i_n}} \leq \frac{1}{\alpha_n} \sum_{x \in \operatorname{Fix}(f^{n+k})} e^{\varphi_n(x) + \gamma_n(\Phi)} \delta_x,$$

where $k = q - 1$. Since

$$\varphi_n \leq \varphi_{n+k} - \varphi_k \circ f^n + C \leq \varphi_{n+k} - \sum_{j=0}^{k-1} \varphi_1 \circ f^{n+j} + kC,$$

we conclude that

$$\bar{\nu}_n \leq \frac{1}{\alpha_n} \sum_{x \in \operatorname{Fix}(f^{n+k})} e^{\varphi_{n+k}(x)+\gamma_n(\Phi)+k(\|\varphi_1\|_\infty + C)} \delta_x \leq \frac{e^\gamma}{\alpha_n} \beta_{n+k} \mu_{n+k}.$$

Moreover, proceeding in a similar manner to that in (10.36) and using (10.18), we obtain

$$\beta_{n+k} \leq \sum_{i_1 \cdots i_{n+k}} a_{i_{n+k} i_1} c_{i_1 \cdots i_{n+k}} \leq \alpha_{n+k} \leq e^C \alpha_n \alpha_k,$$

and hence,

$$\bar{\nu}_n \leq \frac{e^\gamma}{\alpha_n} \beta_{n+k} \mu_{n+k} \leq \frac{e^{\gamma+C}}{\alpha_k} \mu_{n+k}. \tag{10.38}$$

Now we observe that for each $n \in \mathbb{N}$ the measure μ_n is f-invariant. Indeed, given a point $x \in \operatorname{Fix}(f^n)$ corresponding to the repetition of the sequence $(i_1 \cdots i_n)$ in the symbolic dynamics, the set $f^{-1}x$ consists of the points x_j corresponding to the sequences $(j i_1 \cdots i_n i_1 \cdots)$ for which $a_{j i_1} = 1$. But since $f^n(x_j)$ corresponds to the sequence $(i_n i_1 \cdots i_n i_1 \cdots)$, we conclude that $x_j \in \operatorname{Fix}(f^n)$ if and only if $j = i_n$. Hence, $\mu_n(f^{-1}x) = \mu_n(\{x\})$ for every $x \in \operatorname{Fix}(f^n)$. Since the measure μ_n is atomic this shows that it is f-invariant. In particular, any sublimit of the sequence $(\mu_n)_{n\in\mathbb{N}}$ is an f-invariant probability measure in J.

On the other hand, it follows from Theorem 10.1.13 that the sequence of measures $(\nu_n)_{n\in\mathbb{N}}$, or more precisely the sequence $(\bar{\nu}_n)_{n\in\mathbb{N}}$, has a single sublimit which is an invariant Gibbs measure. By the inequalities in (10.37) and (10.38), since any sublimit of the sequence $(\mu_n)_{n\in\mathbb{N}}$ is f-invariant, we conclude that this sequence has in fact a single sublimit. This establishes the convergence of the sequence $(\mu_n)_{n\in\mathbb{N}}$, and since by Theorem 10.1.9 the unique invariant Gibbs measure is also the unique equilibrium measure we obtain the desired statement. □

10.2 Hyperbolic sets

We consider briefly in this section the case of hyperbolic sets, and we describe corresponding results to those for repellers in the former section.

Let Λ be a hyperbolic set for a C^1 diffeomorphism. We always assume in this section that:

1. Λ is locally maximal, that is, $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n U$ for some open set $U \supset \Lambda$;
2. f is topologically mixing on Λ.

Given a Markov partition $R_1, \ldots, R_\kappa$ of Λ, we consider the corresponding $\kappa \times \kappa$ transition matrix $A = (a_{ij})$, and the two-sided topological Markov chain $\sigma|\Sigma_A$. We continue to denote by S_n the set of all Σ_A-admissible sequences of length n, that is, the finite sequences $(i_1 \cdots i_n)$ for which there exists $(\cdots j_0 j_1 j_2 \cdots) \in \Sigma_A$ such that $(i_1 \cdots i_n) = (j_1 \cdots j_n)$. For each $n \in \mathbb{N}$ and $(i_1 \cdots i_n) \in S_n$, we consider again the sets $R_{i_1 \cdots i_n}$ defined by (10.1).

Repeating arguments in the proofs of Theorems 10.1.3 and 10.1.4 (and using the same notation), we obtain formulas for the nonadditive topological pressure of an almost additive sequence.

Theorem 10.2.1 ([6]). *Let Λ be a hyperbolic set for a C^1 diffeomorphism which is topologically mixing on Λ, and let Φ be an almost additive sequence of continuous functions in Λ with tempered variation. Then*

$$\begin{aligned} P_\Lambda(\Phi) &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{i_1\cdots i_n} \exp \varphi_n(x_{i_1\cdots i_n}) \\ &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{x\in \mathrm{Fix}(f^n)} \exp \varphi_n(x) \end{aligned}$$

for any points $x_{i_1\cdots i_n} \in R_{i_1\cdots i_n}$, for each $(i_1 \cdots i_n) \in S_n$ and $n \in \mathbb{N}$.

Proceeding in a similar manner to that for repellers, we also obtain the following results.

Theorem 10.2.2 ([6]). *Let Λ be a hyperbolic set for a C^1 diffeomorphism f which is topologically mixing on Λ, and let Φ be an almost additive sequence of continuous functions in Λ with tempered variation. Then*

$$\begin{aligned} P_\Lambda(\Phi) &= \max_{\mu\in\mathcal{M}_f} \left(h_\mu(f) + \int_\Lambda \lim_{n\to\infty} \frac{\varphi_n(x)}{n}\, d\mu(x) \right) \\ &= \max_{\mu\in\mathcal{M}_f} \left(h_\mu(f) + \lim_{n\to\infty} \frac{1}{n} \int_\Lambda \varphi_n \, d\mu \right), \end{aligned}$$

where the suprema are taken over all f-invariant probability measures μ in Λ.

Theorem 10.2.3 ([6]). *Let Λ be a hyperbolic set for a C^1 diffeomorphism which is topologically mixing on Λ, and let Φ be an almost additive sequence of continuous functions in Λ with bounded variation. Then:*

1. *there is a unique equilibrium measure for Φ;*
2. *there is a unique invariant Gibbs measure for Φ;*
3. *the two measures coincide, are ergodic, and are equal to the limit of the sequence of invariant probability measures μ_n in (10.34).*

Mummert [139] established independently the existence of a unique equilibrium measure under the hypotheses of Theorem 10.2.3.

10.3 General variational principle

We describe in this section how some of the results for repellers and hyperbolic sets can be extended to more general classes of maps.

We first present a variational principle for the topological pressure.

Theorem 10.3.1 ([6]). *Let $f\colon X \to X$ be a continuous transformation of a compact metric space, and let Φ be an almost additive sequence of continuous functions in X with tempered variation. Then*

$$\begin{aligned} P_X(\Phi) &= \sup_{\mu\in\mathcal{M}_f}\left(h_\mu(f) + \int_X \lim_{n\to\infty}\frac{\varphi_n(x)}{n}\,d\mu(x)\right) \\ &= \sup_{\mu\in\mathcal{M}_f}\left(h_\mu(f) + \lim_{n\to\infty}\frac{1}{n}\int_X \varphi_n\,d\mu\right), \end{aligned} \tag{10.39}$$

where the suprema are taken over all f-invariant probability measures μ in Λ.

Proof. To establish the inequality

$$P_X(\Phi) \le \sup_{\mu\in\mathcal{M}_f}\left(h_\mu(f) + \lim_{n\to\infty}\frac{1}{n}\int_X \varphi_n\,d\mu\right), \tag{10.40}$$

one can proceed in a similar manner to that in the proof of Theorem 10.1.5. Namely, replacing the sets $J_{m,u}$ by corresponding sets $X_{m,u}$, we obtain

$$M_{X_{m,u}}(\alpha,\Phi,\mathcal{U}) \le \limsup_{l\to\infty}\sum_{N=l}^\infty \exp\left[N\left(c+4\delta-\alpha+\limsup_{n\to\infty}\frac{\gamma_n(\Phi,\mathcal{U})}{n}\right)\right],$$

where c denotes the supremum in (10.40). Therefore,

$$P_{X_{m,u}}(\Phi,\mathcal{U}) \le c+4\delta+\limsup_{n\to\infty}\frac{\gamma_n(\Phi,\mathcal{U})}{n}.$$

Since

$$X = \bigcup_{m\in\mathbb{N}}\bigcup_{i=1}^r X_{m,u_i},$$

taking the supremum over m and i yields

$$\begin{aligned} P_X(\Phi) &= \sup_{m,i} P_{X_{m,u_i}}(\Phi) \\ &\le c+4\delta+\lim_{\operatorname{diam}\mathcal{U}\to 0}\limsup_{n\to\infty}\frac{\gamma_n(\Phi,\mathcal{U})}{n} = c+4\delta. \end{aligned}$$

Finally, since δ is arbitrary, we conclude that (10.40) holds. The proof of the reverse inequality can be obtained by repeating arguments in the proof of Lemma 4.3.5, and thus it is omitted. □

Mummert [139] established independently the identity

$$P_X(\Phi) = \sup_{\mu\in\mathcal{M}_f}\left(h_\mu(f) + \lim_{n\to\infty}\frac{1}{n}\int_X \varphi_n\,d\mu\right).$$

although under an additional assumption on the sequence Φ.

We also formulate a criterion for the existence of equilibrium measures.

Theorem 10.3.2 ([6]). *Let $f\colon X \to X$ be a continuous transformation of a compact metric space such that the map $\mu \mapsto h_\mu(f)$ is upper semicontinuous. Then there is at least one equilibrium measure for each almost additive sequence Φ of continuous functions in X with tempered variation.*

Proof. It is shown in the proof of Theorem 10.1.5 (see (10.10)) that the map

$$\mu \mapsto \lim_{n\to\infty} \frac{1}{n} \int_X \varphi_n \, d\mu$$

is continuous for each almost additive sequence Φ. Therefore, the map

$$\mu \mapsto h_\mu(f) + \lim_{n\to\infty} \frac{1}{n} \int_X \varphi_n \, d\mu$$

is upper semicontinuous, and hence there exists a measure $\mu \in \mathcal{M}_f$ at which the suprema in (10.39) are attained. □

Mummert [139] established independently the existence of equilibrium measures for an almost additive sequence in the particular case of expansive homeomorphisms. We recall that if f is one-sided or two-sided expansive (see Definitions 2.4.2 and 2.4.5), then the entropy is upper semicontinuous, and hence, by Theorem 10.3.2 there is at least one equilibrium measure for each almost additive sequence. Incidentally, we observe that there exist plenty of transformations without the specification property for which the entropy is still upper semicontinuous. For example, all β-shifts are expansive, and thus the entropy is upper semicontinuous (see [109] for details), but for β in a residual set of full Lebesgue measure (although the complement has full Hausdorff dimension) the corresponding β-shift does not have the specification property (see [172]).

10.4 Regularity of the pressure

The purpose of this section is to describe some regularity properties of the topological pressure of an almost additive sequence. These were obtained by Barreira and Doutor [8] for continuous maps with upper semicontinuous entropy.

We denote by $A(X)$ the family of almost additive sequences of continuous functions with tempered variation. By Theorem 10.3.2, if the map $\mu \mapsto h_\mu(f)$ is upper semicontinuous, then every sequence in $A(X)$ has an equilibrium measure. Let also $E(X) \subset A(X)$ be the family of sequences with a unique equilibrium measure.

Theorem 10.4.1 ([8]). *If $f\colon X \to X$ is a continuous transformation of a compact metric space, and the map $\mu \mapsto h_\mu(f)$ is upper semicontinuous, then the following properties hold:*

1. *given $\Phi \in A(X)$, the function $t \mapsto P_X(\Phi + t\Psi)$ is differentiable at $t = 0$ for every $\Psi \in A(X)$ if and only if $\Phi \in E(X)$; in this case the unique equilibrium measure μ_Φ for Φ is ergodic, and*

$$\frac{d}{dt} P_X(\Phi + t\Psi)|_{t=0} = \lim_{n\to\infty} \int_X \frac{\psi_n}{n}\, d\mu_\Phi; \tag{10.41}$$

2. *for any open set $A \subset \mathbb{R}$, if $\Phi + t\Psi \in E(X)$ for every $t \in A$, then the function $t \mapsto P_X(\Phi + t\Psi)$ is of class C^1 in A.*

Proof. We follow partially arguments in [109]. Let $t \in \mathbb{R}$ and $\Phi, \Psi \in A(X)$. Then $\Phi + t\Psi \in A(X)$, and this sequence has equilibrium measures. Let μ_t and μ_Φ be equilibrium measures respectively for $\Phi + t\Psi$ and Φ. By Theorem 10.3.1, we have

$$\begin{aligned} P_X(\Phi + t\Psi) - P_X(\Phi) &\geq h_{\mu_\Phi}(f) + \lim_{n\to\infty} \int_X \frac{\varphi_n + t\psi_n}{n}\, d\mu_\Phi - P_X(\Phi) \\ &= t \lim_{n\to\infty} \int_X \frac{\psi_n}{n}\, d\mu_\Phi, \end{aligned}$$

and

$$\begin{aligned} P_X(\Phi + t\Psi) - P_X(\Phi) &= P_X(\Phi + t\Psi) - P_X((\Phi + t\Psi) - t\Psi) \\ &\leq P_X(\Phi + t\Psi) - h_{\mu_t}(f) - \lim_{n\to\infty} \int_X \frac{\varphi_n + t\psi_n - t\psi_n}{n}\, d\mu_t \\ &= t \lim_{n\to\infty} \int_X \frac{\psi_n}{n}\, d\mu_t. \end{aligned}$$

We thus obtain the inequalities

$$\begin{aligned} \lim_{n\to\infty} \int_X \frac{\psi_n}{n}\, d\mu_\Phi &\leq \frac{P_X(\Phi + t\Psi) - P_X(\Phi)}{t} \leq \lim_{n\to\infty} \int_X \frac{\psi_n}{n}\, d\mu_t, \\ \lim_{n\to\infty} \int_X \frac{\psi_n}{n}\, d\mu_\Phi &\geq \frac{P_X(\Phi + t\Psi) - P_X(\Phi)}{t} \geq \lim_{n\to\infty} \int_X \frac{\psi_n}{n}\, d\mu_t, \end{aligned} \tag{10.42}$$

respectively for $t > 0$ and $t < 0$.

Now we assume that the function $t \mapsto P_X(\Phi + t\Psi)$ is differentiable at $t = 0$ for every $\Psi \in A(X)$. Let μ_Φ and ν_Φ be two equilibrium measures for Φ. Given a continuous function $\psi\colon X \to \mathbb{R}$, we consider the sequence of functions $\Psi = (\psi_n)_{n\in\mathbb{N}}$ given by

$$\psi_n = \sum_{k=0}^{n-1} \psi \circ f^k$$

for each $n \in \mathbb{N}$. By Birkhoff's ergodic theorem, we have

$$\int_X \psi\, d\mu = \lim_{n\to\infty} \frac{1}{n} \int_X \psi_n\, d\mu$$

for every measure $\mu \in \mathcal{M}_f$. Therefore, since $t \mapsto P_X(\Phi + t\Psi)$ is differentiable at $t = 0$, it follows from (10.42) that

$$\begin{aligned}\int_X \psi \, d\mu_\Phi &= \lim_{n\to\infty} \int_X \frac{\psi_n}{n} \, d\mu_\Phi \\ &= \lim_{t\to 0} \frac{P_X(\Phi + t\Psi) - P_X(\Phi)}{t} \\ &= \lim_{n\to\infty} \int_X \frac{\psi_n}{n} \, d\nu_\Phi = \int_X \psi \, d\nu_\Phi.\end{aligned}$$

Since the function ψ is arbitrary, we conclude that $\mu_\Phi = \nu_\Phi$.

For the converse we start with an auxiliary statement.

Lemma 10.4.2. *The following properties hold:*

1. *if* $\mu_{t_n} \to \mu$ *when* $n \to \infty$*, for some sequence* $t_n \to 0$*, then* μ *is an equilibrium measure for* Φ*;*
2. *if* $\Phi \in E(X)$*, then* μ *is the unique equilibrium measure* μ_Φ *for the sequence* Φ*.*

Proof of the lemma. For a measure μ as in property 1, by Theorem 10.3.1 we have

$$P_X(\Phi) \geq h_\mu(f) + \lim_{m\to\infty} \frac{1}{m} \int_X \varphi_m \, d\mu. \tag{10.43}$$

On the other hand, it follows from the proof of Theorem 10.1.5 (see (10.10)) that the map

$$\mu \mapsto h_\mu(f) + \lim_{n\to\infty} \frac{1}{n} \int_X \varphi_n \, d\mu$$

is upper semicontinuous. Therefore,

$$\begin{aligned}&h_\mu(f) + \lim_{m\to\infty} \frac{1}{m} \int_X \varphi_m \, d\mu \\ &\geq \limsup_{n\to\infty} \left(h_{\mu_{t_n}}(f) + \lim_{m\to\infty} \frac{1}{m} \int_X \varphi_m \, d\mu_{t_n} \right) \\ &= \limsup_{n\to\infty} \left(P_X(\Phi + t_n\Psi) - t_n \lim_{m\to\infty} \frac{1}{m} \int_X \psi_m \, d\mu_{t_n} \right).\end{aligned} \tag{10.44}$$

Moreover

$$\begin{aligned}P_X(\Phi + t_n\Psi) &= \sup\left\{ h_\mu(f) + \lim_{m\to\infty} \frac{1}{m} \int_X (\varphi_m + t_n\psi_m) \, d\mu : \mu \in \mathcal{M}_f \right\} \\ &\geq \sup\left\{ h_\mu(f) + \lim_{m\to\infty} \frac{1}{m} \int_X \varphi_m \, d\mu : \mu \in \mathcal{M}_f \right\} \\ &\quad - \sup\left\{ - \lim_{m\to\infty} \frac{1}{m} \int_X t_n\psi_m \, d\mu : \mu \in \mathcal{M}_f \right\} \\ &= P_X(\Phi) - \sup\left\{ - \lim_{m\to\infty} \frac{1}{m} \int_X t_n\psi_m \, d\mu : \mu \in \mathcal{M}_f \right\}.\end{aligned} \tag{10.45}$$

Since the sequence $(\psi_n)_{n\in\mathbb{N}}$ is almost additive, we have

$$-C(m-1)+\sum_{k=0}^{m-1}\psi_1(f^k(x))\le\psi_m(x)\le\sum_{k=0}^{m-1}\psi_1(f^k(x))+C(m-1)$$

and thus,

$$\|\psi_m\|_\infty\le m(\|\psi_1\|_\infty+C). \tag{10.46}$$

Therefore, there exists $D>0$ such that for every $\mu\in\mathcal{M}_f$ and $m\in\mathbb{N}$ we have

$$\left|\int_X\frac{\psi_m}{m}\,d\mu\right|\le\frac{\|\psi_m\|_\infty}{m}\le D.$$

We conclude that

$$\sup\left\{-\lim_{m\to\infty}\frac{1}{m}\int_X t_n\psi_m\,d\mu:\mu\in\mathcal{M}_f\right\}\le|t_n|D.$$

On other hand, for every $t\in\mathbb{R}$ we have

$$-t\lim_{m\to\infty}\int_X\frac{\psi_m}{m}\,d\mu\ge-|t|D,$$

and it follows from (10.44) and (10.45) that

$$h_\mu(f)+\lim_{m\to\infty}\int_X\frac{\varphi_m}{m}\,d\mu\ge P_X(\Phi)-2|t_n|D.$$

Since $t_n\to0$, we obtain

$$h_\mu(f)+\lim_{m\to\infty}\int_X\frac{\varphi_m}{m}\,d\mu\ge P_X(\Phi),$$

which together with (10.43) yields the identity

$$h_\mu(f)+\lim_{m\to\infty}\int_X\frac{\psi_m}{m}\,d\mu=P_X(\Phi).$$

This establishes the first property in the lemma. The second property follows readily from the first one. □

Now take $\Phi\in E(X)$, $\Psi\in A(X)$, and $t\in\mathbb{R}$. Let μ_Φ be the unique equilibrium measure for Φ, and let μ_t be an equilibrium measure for $\Phi+t\Psi$. By Lemma 10.4.2, we have $\mu_t\to\mu_\Phi$ when $t\to0$, and hence, it follows from (10.42) that

$$\lim_{t\to0}\frac{P_X(\Phi+t\Psi)-P_X(\Phi)}{t}=\lim_{n\to\infty}\frac{1}{n}\int_X\psi_n\,d\mu_\Phi.$$

This establishes the first statement in the theorem, including identity (10.41).

Now we establish the second statement. Let us assume that Φ has a unique equilibrium measure μ_Φ. We show that μ_Φ is ergodic. Otherwise, there would exist an f-invariant measurable set $Y \subset X$ with $0 < \mu_\Phi(Y) < 1$. Now we consider the f-invariant probability measures ν_1 and ν_2 defined by

$$\nu_1(B) = \frac{\mu_\Phi(B \cap Y)}{\mu_\Phi(Y)} \quad \text{and} \quad \nu_2(B) = \frac{\mu_\Phi(B \cap (X \setminus Y))}{\mu_\Phi(X \setminus Y)}$$

for every measurable $B \subset X$. Then

$$\lim_{n\to\infty} \frac{1}{n} \int_Y \varphi_n \, d\mu_\Phi = \mu_\Phi(Y) \lim_{n\to\infty} \frac{1}{n} \int_X \varphi_n \, d\nu_1,$$

and

$$\lim_{n\to\infty} \frac{1}{n} \int_{X\setminus Y} \varphi_n \, d\mu_\Phi = \mu_\Phi(X \setminus Y) \lim_{n\to\infty} \frac{1}{n} \int_X \varphi_n \, d\nu_2.$$

Since μ_Φ is an equilibrium measure for Φ, we obtain

$$\begin{aligned} P_X(\Phi) &= h_{\mu_\Phi}(f) + \lim_{n\to\infty} \frac{1}{n} \int_X \varphi_n \, d\mu_\Phi \\ &= \mu_\Phi(Y) h_{\nu_1}(f) + \mu_\Phi(X \setminus Y) h_{\nu_2}(f) \\ &\quad + \mu_\Phi(Y) \lim_{n\to\infty} \frac{1}{n} \int_X \varphi_n \, d\nu_1 + \mu_\Phi(X \setminus Y) \lim_{n\to\infty} \frac{1}{n} \int_X \varphi_n \, d\nu_2 \\ &\le \max_{i=1,2} \left\{ h_{\nu_i}(f) + \lim_{n\to\infty} \frac{1}{n} \int_X \varphi_n \, d\nu_i \right\} \le P_X(\Phi). \end{aligned}$$

Therefore, at least one of the measures ν_1 or ν_2 is an equilibrium measure for Φ, and thus it must be equal to μ_Φ. But $\nu_i \ne \mu_\Phi$ for $i = 1, 2$. This contradiction shows that μ_Φ is ergodic.

Now let $A \subset \mathbb{R}$ be an open set such that $\Phi + t\Psi \in E(X)$ for every $t \in A$. For each $s \in A$, we have

$$\begin{aligned} \frac{d}{dt} P_X(\Phi + t\Psi)|_{t=s} &= \frac{d}{dt} P_X(\Phi + s\Psi + (t-s)\Psi)|_{t-s=0} \\ &= \lim_{n\to\infty} \frac{1}{n} \int_X \psi_n \, d\mu_{\Phi+s\Psi}, \end{aligned}$$

and the map $t \mapsto P_X(\Phi + t\Psi)$ is differentiable in the set A. The continuity of the derivative follows from the continuity of the map

$$\mu \mapsto \lim_{n\to\infty} \frac{1}{n} \int_X \varphi_n \, d\mu \tag{10.47}$$

together with Lemma 10.4.2. □

Chapter 11

Nonconformal Repellers

We consider in this chapter a class of nonconformal repellers to which one can apply the almost additive thermodynamic formalism developed in Chapter 10. Namely, we consider the class of repellers satisfying a cone condition, which includes for example repellers with a strongly unstable foliation and repellers with a dominated splitting. In particular, we are interested in the entropy spectrum of the Lyapunov exponents of a repeller. We note that while in the conformal case the multifractal analysis of the Lyapunov exponents is very well understood, quite the contrary happens in the nonconformal case. At least for the class of nonconformal repellers satisfying a cone condition it is still possible to develop substantially a corresponding multifractal analysis. This amounts to verifying that some sequences related to the Lyapunov exponents are almost additive and have bounded variation, which then allows us to apply the theory developed in the former chapter. For simplicity of the exposition, we only consider transformations in the plane.

11.1 Repellers and Lyapunov exponents

We present in this section several basic notions that are needed in the formulation of our main results. In particular, we recall some notions from the theory of Lyapunov exponents. We also consider a model class of nonconformal repellers in $\mathbb{R}^2$ instead of striving for any formal generalization. This will allow us to highlight the main ideas without accessory technicalities.

We first recall some basic notions from the theory of Lyapunov exponents. Given a differentiable transformation $f\colon M \to M$, for each $x \in M$ and $v \in T_xM$ we define the *Lyapunov exponent* of (x, v) (with respect to f) by

$$\chi(x, v) = \limsup_{n\to\infty} \frac{1}{n} \log\|d_x f^n v\|, \tag{11.1}$$

with the convention that $\log 0 = -\infty$. By the abstract theory of Lyapunov exponents (see for example [14]), for each $x \in M$ there exist a positive integer

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$s(x) \le \dim M$, numbers $\chi_1(x) > \cdots > \chi_{s(x)}(x)$, and linear subspaces

$$\{0\} = E_{s(x)+1}(x) \subset E_{s(x)}(x) \subset \cdots \subset E_1(x) = T_x M$$

such that

$$E_i(x) = \big\{v \in T_x M \colon \chi(x,v) \le \chi_i(x)\big\},$$

and $\chi(x,v) = \chi_i(x)$ whenever $v \in E_i(x) \setminus E_{i+1}(x)$, for $i = 1, \ldots, s(x)$. Moreover, it follows from the noninvertible version of Oseledets' multiplicative ergodic theorem (see for example [15]), that for each finite f-invariant measure μ in M there exists a set $X \subset M$ of full μ-measure such that if $x \in X$, then

$$\lim_{n\to\infty} \frac{1}{n} \log\|d_x f^n v\| = \chi_i(x)$$

for every $v \in E_i(x) \setminus E_{i+1}(x)$ and $i = 1, \ldots, s(x)$, with uniform convergence in v on each subspace $F \subset E_i(x)$ such that $F \cap E_{i+1}(x) = \{0\}$ (in particular, the $\limsup$ in (11.1) is now a limit).

When $M = \mathbb{R}^2$, for each $x \in \mathbb{R}^2$ either $s(x) = 1$, and we set

$$\lambda_1(x) = \chi_1(x) \quad \text{and} \quad \lambda_2(x) = \chi_1(x),$$

or $s(x) = 2$, and we set

$$\lambda_1(x) = \chi_1(x) \quad \text{and} \quad \lambda_2(x) = \chi_2(x).$$

In other words, the numbers $\lambda_1(x)$ and $\lambda_2(x)$ are the values of the Lyapunov exponent $\lambda(x,\cdot)$ counted with multiplicities.

Now let J be a repeller of a C^1 map $f \colon \mathbb{R}^2 \to \mathbb{R}^2$. We shall introduce certain sequences of functions obtained from the singular values of the matrices $d_x f^n$. We recall that the *singular values* $\sigma_1(A) \ge \sigma_2(A)$ of a 2×2 matrix A are the eigenvalues, counted with multiplicities, of the matrix $(A^*A)^{1/2}$, where A^* denotes the transpose of A. These are given by

$$\sigma_1(A) = \|A\| \quad \text{and} \quad \sigma_2(A) = \|A^{-1}\|^{-1}$$

when $\mathbb{R}^2$ is equipped with the norm $\|(x_1,x_2)\| = \sqrt{x_1^2 + x_2^2}$. Now we define sequences of functions $\Phi_i = (\varphi_{i,n})_{n\in\mathbb{N}}$ in the repeller J by

$$\varphi_{i,n}(x) = \log \sigma_i(d_x f^n), \tag{11.2}$$

for $n \in \mathbb{N}$ and $i = 1, 2$. We note that each function $\varphi_{i,n}$ is continuous. These sequences are related to the Lyapunov exponents. Indeed, again it follows from Oseledets' multiplicative ergodic theorem that for each finite f-invariant measure μ in $\mathbb{R}^2$ there exists a set $X \subset \mathbb{R}^2$ of full μ-measure such that

$$\lim_{n\to\infty} \frac{\varphi_{i,n}(x)}{n} = \lim_{n\to\infty} \frac{1}{n} \log \sigma_i(d_x f^n) = \lambda_i(x)$$

for every $x \in X$ and $i = 1, 2$.

Given $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$, we consider the level sets

$$E(\alpha) = \left\{ x \in J : \lim_{n\to\infty} \frac{\varphi_n(x)}{n} = \alpha \right\}, \tag{11.3}$$

where $\varphi_n = (\varphi_{1,n}, \varphi_{2,n})$ for each $n \in \mathbb{N}$. One may think of $E(\alpha)$ as a level set of the Lyapunov exponents, and we shall refer to the function $\alpha \mapsto h(f|E(\alpha))$ as the *entropy spectrum of the Lyapunov exponents.*

11.2 Cone condition

We show in this section that for repellers satisfying a cone condition the functions Φ_1 and Φ_2 defined by (11.2) are almost additive. This will allow us to apply the almost additive formalism developed in Chapter 10.

Given $\gamma \le 1$ and a 1-dimensional subspace $E(x) \subset T_x\mathbb{R}^2$, we define the *cone*

$$C_\gamma(x) = \big\{ (u, v) \in E(x) \oplus E(x)^\perp : \|v\| \le \gamma \|u\| \big\}. \tag{11.4}$$

Definition 11.2.1. A differentiable map $f \colon \mathbb{R}^2 \to \mathbb{R}^2$ is said to satisfy a *cone condition* on a set $J \subset \mathbb{R}^2$ if there exist $\gamma \le 1$ and for each $x \in J$ a 1-dimensional subspace $E(x) \subset T_x\mathbb{R}^2$ varying continuously with x such that

$$(d_x f) C_\gamma(x) \subset \{0\} \cup \operatorname{int} C_\gamma(fx). \tag{11.5}$$

We present several examples of maps satisfying a cone condition.

Example 11.2.2. Let us assume that for each $x \in J$ the derivative $d_x f$ is a positive 2×2 matrix. Then the first quadrant Q is invariant under $d_x f$, that is, $(d_x f)Q \subset Q$ for each $x \in J$. Therefore, the map f satisfies the cone condition in (11.5) with $\gamma = 1$, taking for $E(x)$ the 1-dimensional subspace making an angle of $\pi/4$ with the horizontal direction. This example is related to work of Feng and Lau in [68] (see also [64]).

Another class of examples corresponds to the existence of a strongly unstable foliation.

Example 11.2.3. Let J be a locally maximal repeller, in the sense that J has some open neighborhood U where it is the only invariant set. In this case $f^{-1}J \cap U = J$. We assume that there exists a *strongly unstable foliation* of the set U, that is, a foliation by 1-dimensional C^2 leaves $V(x)$ such that:

1. $f(V(x)) \supset V(fx)$ for every $x \in U \cap f^{-1}U$;
2. there exist constants $c > 0$ and $\lambda \in (0, 1)$ such that
$$\frac{|\det d_x f^n|}{\|d_x f^n | T_x V(x)\|^2} \le c\lambda^n$$
for every $n \in \mathbb{N}$ and $x \in \bigcap_{i=0}^n f^{-i} U$.

It was shown by Hu in [92] that this assumption is equivalent to the following two requirements:

1. for some choice of subspaces $E(x)$ varying continuously with x, the cone condition in (11.5) holds for every $x \in U \cap f^{-1}U$;
2. there exist 1-dimensional subspaces $F(x) \subset \{0\} \cup \operatorname{int} C_\gamma(x)$ for $x \in U \cap f^{-1}U$ such that $d_x f F(x) = F(f(x))$.

Hence, repellers with a strongly unstable foliation satisfy a cone condition.

We note that the cone condition in (11.5) is weaker than assuming the existence of a strongly unstable foliation. In particular, condition (11.5) does not ensure the existence of an invariant distribution $F(x)$ as in Example 11.2.3. On the other hand, when there exists a strongly unstable foliation the invariant distribution $F(x)$ is given by (see [92])

$$F(x) = \bigcap_{n\in\mathbb{N}} \bigcup_{y\in f^{-n}x} d_y f^n C_\gamma(y).$$

It is thus independent of the particular preimages $x_n \in f^{-n}x$, that is,

$$F(x) = \bigcap_{n\in\mathbb{N}} d_{x_n} f^n C_\gamma(x_n).$$

One can also consider repellers with a dominated splitting.

Example 11.2.4. We say that a repeller J has a *dominated splitting* if there exists a decomposition $T_J\mathbb{R}^2 = E \oplus F$ such that:

1. $d_x f E(x) = E(f(x))$ and $d_x f F(x) = F(f(x))$ for every $x \in J$;
2. there exist constants $c > 0$ and $\lambda \in (0,1)$ such that

$$\|d_x f^n | E\| \cdot \|(d_x f)^{-n} | F\| \le c\lambda^n$$

   for every $x \in J$ and $n \in \mathbb{N}$.

It follows easily from the definition that the subspaces $E(x)$ and $F(x)$ vary continuously with x. Furthermore, one can verify that when there exists a dominated splitting of J the map f satisfies a cone condition on J.

We note that the existence of a strongly unstable foliation does not ensure the existence of a dominated splitting, due to the requirement of a df-invariant decomposition $E \oplus F$ (more precisely, the existence of a strongly unstable foliation only ensures the existence of the invariant distribution F in Example 11.2.3).

The following result of Barreira and Gelfert shows that the sequences of functions in (11.2) are almost additive when the cone condition in (11.5) holds.

Theorem 11.2.5 ([10]). *Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 local diffeomorphism, and let $J \subset \mathbb{R}^2$ be a compact f-invariant set. If f satisfies a cone condition on J, then there exists $C \geq 1$ such that*

$$C^{-1}\sigma_1(d_x f^n)\sigma_1(d_{f^n(x)} f^m) \leq \sigma_1(d_x f^{n+m}) \leq \sigma_1(d_x f^n)\sigma_1(d_{f^n(x)} f^m),$$

and

$$\sigma_2(d_x f^n)\sigma_2(d_{f^n(x)} f^m) \leq \sigma_2(d_x f^{n+m}) \leq C\sigma_2(d_x f^n)\sigma_2(d_{f^n(x)} f^m)$$

for every $x \in J$ and $n, m \in \mathbb{N}$.

Proof. Let $x \in J$. We note that $C_\gamma(x)$ is taken by the linear transformation $d_x f$ into a subset of $C_\gamma(f(x))$ that is also a cone, namely

$$d_x f C_\gamma(x) = \bigl\{(u, v) \in F(x) \oplus F(x)^\perp : \|v\| \leq \gamma(x)\|u\|\bigr\}$$

for some number $\gamma(x) < 1$ and some 1-dimensional subspace $F(x) \subset T_{fx}\mathbb{R}^2$. Furthermore, since f is of class C^1 and the subspaces $E(x)$ in (11.4) vary continuously with x, we can always assume that the function $x \mapsto \gamma(x)$ is continuous.

Given unit vectors $v, u_1 \in C_\gamma(x)$, let u_2 be a unit vector orthogonal to u_1 such that u_1 and u_2 are eigenvectors of the matrix $((d_x f)^* d_x f)^{1/2}$. Then the vectors $v_1 = d_x f u_1$ and $v_2 = d_x f u_2$ are also orthogonal. Writing $v = \cos\beta_v u_1 + \sin\beta_v u_2$, we have

$$|\cos\beta_v| = \cos\angle(u_1, v) \geq \frac{1 - \gamma(x)^2}{1 + \gamma(x)^2} > 0, \tag{11.6}$$

and by the continuity of the function $x \mapsto \gamma(x)$,

$$a := \inf_{x \in J} \frac{1 - \gamma(x)^2}{1 + \gamma(x)^2} > 0.$$

Since

$$d_x f v = \cos\beta_v d_x f u_1 + \sin\beta_v d_x f u_2 = \cos\beta'_v \frac{v_1}{\|v_1\|} + \sin\beta'_v \frac{v_2}{\|v_2\|}, \tag{11.7}$$

it follows from (11.6) that

$$\|d_x f v\| \geq |\cos\beta_v| \cdot \|d_x f u_1\| \geq a\|d_x f u_1\|. \tag{11.8}$$

On the other hand, by (11.7), we have

$$\tan\beta'_v = \tan\beta_v \|v_2\|/\|v_1\|. \tag{11.9}$$

Now let $v, w \in C_\gamma(x) \setminus \operatorname{int} C_\gamma(x)$ be vectors with positive projection in the direction of u_1. By the cone condition in (11.5), we have

$$\beta_v + \beta_w = \angle(v, w) > \angle(d_x f v, d_x f w) = \beta'_v + \beta'_w. \tag{11.10}$$

If $\|v_2\| \geq \|v_1\|$, then it would follow from (11.9) that $\beta'_v \geq \beta_v$ and $\beta'_w \geq \beta_w$. But this contradicts (11.10). Therefore, we must have $\|v_2\| < \|v_1\|$, and

$$\|v_i\| = \|d_x f u_i\| = \sigma_i(d_x f) \quad \text{for} \quad i = 1, 2. \tag{11.11}$$

Given $y \in J$ and $k \in \mathbb{N}$, let $v_{y,k} \neq 0$ be an eigenvector of the 2×2 matrix $((d_y f^k)^* d_y f^k)^{1/2}$ corresponding to the largest eigenvalue. It follows from (11.11) (with f replaced by f^k) that $v_{y,k} \in C_\gamma(y)$. Now we consider the vector

$$\widehat{w} = d_x f^n v_{x,n+m} / \|d_x f^n v_{x,n+m}\| \in C_\gamma(f^n x).$$

Since

$$\|d_x f^n v_{x,n}\| = \|d_x f^n\| \quad \text{and} \quad \|d_x f^{n+m} v_{x,n+m}\| = \|d_x f^{n+m}\|,$$

it follows from (11.8) (with f replaced by f^n and f^m) that

$$\|d_x f^n v_{x,n+m}\| \geq a \|d_x f^n v_{x,n}\| \quad \text{and} \quad \|d_{f^n x} f^m \widehat{w}\| \geq a \|d_{f^n x} f^m v_{f^n x, m}\|.$$

Using the identity $d_x f^{n+m} = d_{f^n x} f^m \circ d_x f^n$, we obtain

$$\begin{aligned} \|d_x f^{n+m}\| &= \|d_{f^n x} f^m \widehat{w}\| \cdot \|d_x f^n v_{x,n+m}\| \\ &\geq a^2 \|d_{f^n x} f^m v_{f^n x, m}\| \cdot \|d_x f^n v_{x,n}\|. \end{aligned}$$

This establishes the first statement in the theorem. The second statement follows readily from the identities $|\det A| = \sigma_1(A)\sigma_2(A)$ and $|\det(AB)| = |\det A| \cdot |\det B|$. $\square$

By Theorem 11.2.5, the sequences of functions Φ_1 and Φ_2 defined by (11.2) are almost additive. More precisely, the following statement holds.

Theorem 11.2.6. *Let J be a repeller of a C^1 map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$. If f satisfies a cone condition on J, then the sequences Φ_1 and Φ_2 are almost additive.*

11.3 Bounded distortion

We introduce in this section the notion of bounded distortion for a repeller. This condition ensures that the sequences Φ_i defined by (11.2) have bounded variation.

Let again J be a repeller of a C^1 map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$. Take $\delta > 0$ such that f is invertible on the ball $B(x, \delta)$ for every $x \in J$ (simply take a Lebesgue number of a cover by open balls such that f is invertible on each of them). For each $x \in J$ and $n \in \mathbb{N}$, we define

$$B_n(x, \delta) = \bigcap_{l=0}^{n-1} f^{-l} B(f^l(x), \delta).$$

We always assume in this chapter that the diameter of the Markov partition used to define the sets $R_{i_1 \cdots i_n}$ in (10.1) is at most $\delta/2$ (we recall that any repeller has Markov partitions of arbitrarily small diameter). This ensures that

$$R_{i_1 \cdots i_n} \subset B_n(x, \delta) \text{ for every } x = \chi(i_1 i_2 \cdots) \in J \text{ and } n \in \mathbb{N}, \tag{11.12}$$

where χ is the coding map of the repeller.

Definition 11.3.1. We say that f has *tempered distortion* on J if for some $\delta > 0$,

$$\limsup_{n \to \infty} \frac{1}{n} \log \sup \left\{ \|d_y f^n (d_z f^n)^{-1}\| : x \in J \text{ and } y, z \in B_n(x, \delta) \right\} = 0, \tag{11.13}$$

and that f has *bounded distortion* on J if for some $\delta > 0$,

$$\sup \left\{ \|d_y f^n (d_z f^n)^{-1}\| : x \in J \text{ and } y, z \in B_n(x, \delta) \right\} < \infty.$$

We note that if any of these properties holds for some δ then it also holds for any smaller δ.

Now we give a condition for the bounded distortion of a $C^{1+\alpha}$ transformation in terms of the notion of bunching in Definition 5.2.6. The following statement is a simple consequence of Proposition 5.2.7.

Proposition 11.3.2. *Let J be a repeller of a $C^{1+\alpha}$ map $f \colon \mathbb{R}^2 \to \mathbb{R}^2$ which is α-bunched on J. Then f has bounded distortion on J.*

We notice that any conformal map is α-bunched for every $\alpha > 0$.

We also present a criterion for the bounded variation of the sequences Φ_i defined by (11.2).

Proposition 11.3.3. *Let J be a repeller of a C^1 map $f \colon \mathbb{R}^2 \to \mathbb{R}^2$. If f has bounded distortion on J, then the sequences Φ_1 and Φ_2 have bounded variation.*

This is a particular case of a more general statement. For the formulation of the result, we write

$$\langle d, \varphi_n \rangle = \sum_{i=1}^{2} d_i \varphi_{i,n} \quad \text{for} \quad d = (d_1, d_2) \in \mathbb{R}^2,$$

where $\varphi_n = (\varphi_{1,n}, \varphi_{2,n})$ is again the sequence of functions defined by (11.2).

Proposition 11.3.4 ([10]). *Let J be a repeller of a C^1 map $f \colon \mathbb{R}^2 \to \mathbb{R}^2$.*

1. *If f has tempered distortion on J, then there exists a sequence $(\rho_n)_{n \in \mathbb{N}} \subset \mathbb{R}^+$ decreasing to 0 such that*

$$\frac{\max_{x \in R_{i_1 \cdots i_n}} \exp\langle d, \varphi_n(x) \rangle}{\min_{y \in R_{i_1 \cdots i_n}} \exp\langle d, \varphi_n(y) \rangle} \le e^{n \rho_n \|d\|} \tag{11.14}$$

for every $d \in \mathbb{R}^2$, $n \in \mathbb{N}$, and $(i_1 \cdots i_n) \in S_n$.

2. *If f has bounded distortion on J, then there exists $D > 0$ such that*

$$\frac{\max_{x \in R_{i_1 \cdots i_n}} \exp\langle d, \varphi_n(x)\rangle}{\min_{y \in R_{i_1 \cdots i_n}} \exp\langle d, \varphi_n(y)\rangle} \le D^{\|d\|} \tag{11.15}$$

for every $d \in \mathbb{R}^2$, $n \in \mathbb{N}$, and $(i_1 \cdots i_n) \in S_n$.

Proof. We first assume that f has tempered distortion on J. By (11.13), there exists a sequence $(\rho_n)_{n\in\mathbb{N}} \subset \mathbb{R}^+$ decreasing to zero such that

$$\|d_y f^n (d_z f^n)^{-1}\| \le e^{n\rho_n/2} \text{ for every } x \in J \text{ and } y, z \in B_n(x, \delta).$$

Since $d_y f^n = (d_y f^n (d_z f^n)^{-1}) d_z f^n$, this yields

$$d_y f^n(B) \subset e^{n\rho_n/2} d_z f^n(B),$$

where $B \subset T_x M$ is the unit ball centered at 0. Since the numbers $\sigma_i(d_y f^n)$ coincide with the lengths of the semiaxes of $d_y f^n(B)$ we conclude that

$$\sigma_i(d_y f^n)/\sigma_i(d_z f^n) \le e^{n\rho_n},$$

and thus also $\varphi_{i,n}(y) - \varphi_{i,n}(z) \le n\rho_n$, for $i = 1, 2$. Therefore,

$$\langle d, \varphi_n(y)\rangle - \langle d, \varphi_n(z)\rangle \le n\rho_n \|d\|,$$

and in view of (11.12) this yields inequality (11.14). When the map f has bounded distortion on J one can replace $n\rho_n$ by a constant, and thus we obtain inequality (11.15). □

11.4 Cone condition plus bounded distortion

Now we apply the almost additive thermodynamic formalism to the class of repellers satisfying a cone condition for maps with bounded distortion, as a simple consequence of the results in the former sections.

We consider the sequence $\langle d, \Phi\rangle = (\langle d, \varphi_n\rangle)_{n\in\mathbb{N}}$, where $\Phi = (\Phi_1, \Phi_2)$.

Definition 11.4.1. A vector $\alpha \in \mathbb{R}^2$ is said to be a *gradient* of the topological pressure, or more precisely of the function $Q\colon d \mapsto P_J(\langle d, \Phi\rangle)$, if there exists $d \in \mathbb{R}^2$ such that $\alpha = \nabla Q(d)$ (in particular, this includes the requirement that the gradient is well defined).

We note that the vector $d = d(\alpha)$ may not be unique. We shall show that the function Q is convex (see Lemma 11.6.5). Thus, it follows readily from convex analysis (see for example [163]) that Q is differentiable in a G_δ dense set.

By Theorem 11.2.6 and Proposition 11.3.3, if the map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ satisfies a cone condition on J and has bounded distortion on J, then the sequences Φ_1 and Φ_2 are almost additive sequences. This allows us to apply the results in Chapter 10. In particular, the following result is an immediate consequence of Theorems 10.1.5, 10.1.9, and 10.1.14.

Theorem 11.4.2. *Let J be a repeller of a C^1 map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ which is topologically mixing on J. If f satisfies a cone condition on J, and f has bounded distortion on J, then for $i = 1, 2$ the following properties hold:*

1. *the topological pressure satisfies the variational principle*

$$\begin{aligned} P_J(\Phi_i) &= \max_{\mu \in \mathcal{M}_f} \left(h_\mu(f) + \int_J \lambda_i(x)\, d\mu(x) \right) \\ &= \max_{\mu \in \mathcal{M}_f} \left(h_\mu(f) + \lim_{n\to\infty} \frac{1}{n} \int_J \log \sigma_i(d_x f^n)\, d\mu(x) \right), \end{aligned}$$

   where the suprema are taken over all f-invariant probability measures μ in J;

2. *there is a measure $\mu_i \in \mathcal{M}_f$ which is the unique equilibrium measure for Φ_i and the unique invariant Gibbs measure for Φ_i;*

3. *there exists a constant $K > 0$ such that*

$$K^{-1} \le \frac{\mu_i(R_{i_1\cdots i_n})}{\exp[-nP_J(\Phi_i)]\sigma_i(d_x f^n)} \le K$$

   for every $n \in \mathbb{N}$, $(i_1 \cdots i_n) \in S_n$, and $x \in R_{i_1\cdots i_n}$;

4. *the measure μ_i is ergodic, and*

$$\sum_{x\in \mathrm{Fix}(f^n)} \sigma_i(d_x f^n)\delta_x \Big/ \sum_{x\in \mathrm{Fix}(f^n)} \sigma_i(d_x f^n) \to \mu_i \quad \text{when} \quad n \to \infty.$$

11.5 Construction of weak Gibbs measures

For repellers satisfying a cone condition and having tempered distortion (but not necessarily bounded distortion as in Theorem 11.4.2), we describe a family of probability measures in the repeller, constructed by Barreira and Gelfert, that satisfy a weak version of the Gibbs property. These measures are ergodic, although a priori not necessarily invariant, and they are crucial in the multifractal analysis of nonconformal repellers.

Theorem 11.5.1 ([10]). *Let J be a repeller of a C^1 map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ which is topologically mixing on J, such that:*

1. *f satisfies a cone condition on J;*
2. *f has tempered distortion on J.*

Then there exist a constant $K > 0$, a sequence $(\rho_n)_{n\in\mathbb{N}} \subset \mathbb{R}^+$ decreasing to 0, and for each $d \in \mathbb{R}^2$ an ergodic measure ν_d in J such that

$$K^{-(1+\|d\|)} e^{-n\rho_n\|d\|} \le \frac{\nu_d(R_{i_1\cdots i_n})}{\exp[-nP_J(\langle d, \Phi\rangle) + \langle d, \varphi_n(x)\rangle]} \le K^{1+\|d\|} e^{n\rho_n\|d\|} \quad (11.16)$$

for every $n \in \mathbb{N}$, $(i_1 \cdots i_n) \in S_n$, and $x \in R_{i_1\cdots i_n}$.

Proof. Let

$$\begin{aligned}\Phi_n(i_1\cdots i_n,d) &= \max\left\{\exp\langle d,\varphi_n(y)\rangle : y\in R_{i_1\cdots i_n}\right\}\\ &= \max\left\{\sigma_1(d_yf^n)^{d_1}\sigma_2(d_yf^n)^{d_2} : y\in R_{i_1\cdots i_n}\right\},\end{aligned}\tag{11.17}$$

with the convention that $\Phi_n(i_1\cdots i_n,d)=0$ when $R_{i_1\cdots i_n}=\varnothing$. Let also

$$\Phi_n(d)=\sum_{i_1\cdots i_n}\Phi_n(i_1\cdots i_n,d).$$

In the following lemmas we use the almost multiplicativity of the singular values in Theorem 11.2.5 to obtain the almost multiplicativity of the functions Φ_n.

We note that since f is topologically mixing on J, there exists $q\in\mathbb{N}$ such that A^q has only positive entries. This ensures that given sequences $(i_1\cdots i_n)\in S_n$ and $(j_1\cdots j_l)\in S_l$ there exists $(p_1\cdots p_k)\in S_k$ with $k=q-1$ such that

$$(i_1\cdots i_n p_1\cdots p_k j_1\cdots j_l)\in S_{n+k+l}.$$

Lemma 11.5.2. *There exists $K_1>0$ such that for every $d\in\mathbb{R}^2$ and $l>k$ we have*

$$K_1^{-\|d\|}\Phi_{l-k}(d)\le\Phi_l(d)\le p^kK_1^{\|d\|}\Phi_{l-k}(d).$$

Proof of the lemma. Given $y\in R_{i_1\cdots i_l}$ such that

$$\exp\langle d,\varphi_l(y)\rangle=\Phi_l(i_1\cdots i_l,d),\tag{11.18}$$

it follows from Theorem 11.2.5 that

$$\begin{aligned}\Phi_l(i_1\cdots i_l,d) &\le C^{\|d\|}\exp\langle d,\varphi_k(y)\rangle\exp\langle d,\varphi_{l-k}(f^k(y))\rangle\\ &\le C^{\|d\|}C'^{k\|d\|}\Phi_{l-k}(j_1\cdots j_{l-k},d),\end{aligned}\tag{11.19}$$

for some constant $C'>0$. Furthermore, for each $x\in R_{i_1\cdots i_l}$ we have

$$\begin{aligned}\Phi_l(i_1\cdots i_l,d) &\ge \exp\langle d,\varphi_l(x)\rangle\\ &\ge C^{-\|d\|}\exp\langle d,\varphi_k(x)\rangle\exp\langle d,\varphi_{l-k}(f^k(x))\rangle.\end{aligned}\tag{11.20}$$

If in addition x satisfies the identity

$$\exp\langle d,\varphi_{l-k}(f^k(x))\rangle=\Phi_{l-k}(j_1\cdots j_{l-k},d),$$

then it follows from (11.19) and (11.20) that

$$\begin{aligned}C^{\|d\|}C'^{k\|d\|}\Phi_{l-k}(j_1\cdots j_{l-k},d) &\ge \Phi_l(i_1\cdots i_l,d)\\ &\ge C^{-\|d\|}C'^{-k\|d\|}\Phi_{l-k}(j_1\cdots j_{l-k},d),\end{aligned}$$

by eventually enlarging the constant C'. On the other hand, since A^q has only positive entries, for each $(j_1 \cdots j_{l-k}) \in S_{l-k}$ there exists $(n_1 \cdots n_k) \in S_k$ with $k = q-1$ such that

$$(n_1 \cdots n_k j_1 \cdots j_{l-k}) \in S_l.$$

Therefore,

$$\begin{aligned} &p^k C^{\|d\|} C'^{k\|d\|} \sum_{j_1 \cdots j_{l-k}} \Phi_{l-k}(j_1 \cdots j_{l-k}, d) \\ &\geq \sum_{i_1 \cdots i_l} \Phi_l(i_1 \cdots i_l, d) \geq C^{-\|d\|} C'^{-k\|d\|} \sum_{j_1 \cdots j_{l-k}} \Phi_{l-k}(j_1 \cdots j_{l-k}, d). \end{aligned}$$

This completes the proof of the lemma □

Lemma 11.5.3. *There exists $K_2 > 0$ such that for every $d \in \mathbb{R}^2$, $n \in \mathbb{N}$, $(i_1 \cdots i_n) \in S_n$, and $l > k$ we have*

$$\sum_{j_1 \cdots j_l} \Phi_{n+l}(i_1 \cdots i_n j_1 \cdots j_l, d) \leq C^{\|d\|} \Phi_n(i_1 \cdots i_n, d) \Phi_l(d)$$

and

$$\sum_{j_1 \cdots j_l} \Phi_{n+l}(i_1 \cdots i_n j_1 \cdots j_l, d) \geq \Phi_n(i_1 \cdots i_n, d) \Phi_l(d) e^{-n\rho_n \|d\|} p^{-k} K_2^{-\|d\|}.$$

Proof of the lemma. Take $(i_1 \cdots i_n) \in S_n$ and choose $(j_1 \cdots j_l) \in S_l$ such that $(i_1 \cdots i_n j_1 \cdots j_l) \in S_{n+l}$. Given $y \in R_{i_1 \cdots i_l}$ satisfying (11.18), it follows from Theorem 11.2.5 that

$$\Phi_{n+l}(i_1 \cdots i_n j_1 \cdots j_l, d) \leq C^{\|d\|} \Phi_n(i_1 \cdots i_n, d) \Phi_l(j_1 \cdots j_l, d). \tag{11.21}$$

Therefore,

$$\sum_{j_1 \cdots j_l} \Phi_{n+l}(i_1 \cdots i_n j_1 \cdots j_l, d) \leq C^{\|d\|} \Phi_n(i_1 \cdots i_n, d) \Phi_l(d),$$

which establishes the first inequality in the lemma.

Now we observe that for each $(j_1 \cdots j_{l-k}) \in S_{l-k}$ there exists $(m_1 \cdots m_k) \in S_k$ such that

$$(i_1 \cdots i_n m_1 \cdots m_k j_1 \cdots j_{l-k}) \in S_{n+l}.$$

Hence, for each $x \in R_{i_1 \cdots i_n m_1 \cdots m_k j_1 \cdots j_{l-k}}$ we have

$$\begin{aligned} &\Phi_{n+l}(i_1 \cdots i_n k_1 \cdots k_k j_1 \cdots j_{l-k}, d) \\ &\geq C^{-2\|d\|} \exp\langle d, \varphi_n(x)\rangle \exp\langle d, \varphi_k(f^n(x))\rangle \exp\langle d, \varphi_{l-k}(f^{n+k}(x))\rangle. \end{aligned} \tag{11.22}$$

If in addition x satisfies the identity

$$\exp\langle d, \varphi_{l-k}(f^{n+k}(x))\rangle = \Phi_{l-k}(j_1 \cdots j_{l-k}, d),$$

then it follows from (11.22) and Proposition 11.3.4 that

$$\begin{aligned}
&\Phi_{n+l}(i_1\cdots i_n m_1\cdots m_k j_1\cdots j_{l-k},d)\\
&\geq C^{-2\|d\|}C'^{-k\|d\|}\exp\langle d,\varphi_n(x)\rangle\exp\langle d,\varphi_{l-k}(f^{n+k}(x))\rangle\\
&\geq C^{-2\|d\|}C'^{-k\|d\|}\Phi_n(i_1\cdots i_n,d)e^{-n\rho_n\|d\|}\Phi_{l-k}(j_1\cdots j_{l-k},d),
\end{aligned}\tag{11.23}$$

for some constant $C'>0$. Summing over all sequences in S_l, it follows from Lemma 11.5.2 that

$$\begin{aligned}
&\sum_{t_1\cdots t_l}\Phi_{n+l}(i_1\cdots i_n t_1\cdots t_l,d)\\
&\geq\sum_{j_1\cdots j_{l-k}}\Phi_{n+l}(i_1\cdots i_n m_1\cdots m_k j_1\cdots j_{l-k},d)\\
&\geq C^{-2\|d\|}C'^{-k\|d\|}\Phi_n(i_1\cdots i_n,d)e^{-n\rho_n\|d\|}\Phi_{l-k}(d)\\
&\geq C^{-2\|d\|}C'^{-k\|d\|}\Phi_n(i_1\cdots i_n,d)e^{-n\rho_n\|d\|}p^{-k}K_1^{-\|d\|}\Phi_l(d).
\end{aligned}\tag{11.24}$$

This establishes the second inequality in the lemma. □

The following is an immediate consequence of Lemma 11.5.3.

Lemma 11.5.4. *For every $d\in\mathbb{R}^2$, $n\in\mathbb{N}$, and $l>k$ we have*

$$\Phi_l(d)\Phi_n(d)e^{-n\rho_n\|d\|}p^{-k}K_2^{-\|d\|}\leq\Phi_{l+n}(d)\leq C^{\|d\|}\Phi_l(d)\Phi_n(d).$$

To establish the almost additivity of the sequence $n\mapsto\log\Phi_n(d)$, we first establish an auxiliary result.

Lemma 11.5.5. *Let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence of real numbers satisfying*

$$\xi_{n+m}\geq\xi_n\xi_m e^{-n\rho_n}\quad\textit{for every}\quad n,m\in\mathbb{N},$$

with $\rho_n\to 0$ when $n\to\infty$. Then the limit $\lim_{n\to\infty}\xi_n^{1/n}$ exists, and

$$\lim_{n\to\infty}\xi_n^{1/n}\geq\sup_{n\geq1}(\xi_n^{1/n}e^{-\rho_n}).$$

Proof of the lemma. Each integer $q\in\mathbb{N}$ can be written in the form $q=kp+l$, with $k\in\mathbb{N}$ and $0\leq l<p$. By hypothesis, we have $\xi_q\geq\xi_p^k\xi_l e^{-kp\rho_p}$. Since $kp\leq q$, we obtain

$$\xi_q^{1/q}\geq(\xi_p^{1/p}e^{-\rho_p})^{kp/q}\min_{0\leq l<p}\xi_l^{1/q}.$$

Letting $q\to\infty$, with p fixed, we obtain $kp/q\to 1$, and hence,

$$\liminf_{q\to\infty}\xi_q^{1/q}\geq\xi_p^{1/p}e^{-\rho_p}.$$

This yields

$$\begin{aligned}\liminf_{q\to\infty} \xi_q^{1/q} &\ge \sup_{p\ge 0} \xi_p^{1/p} e^{-\rho_p} \\ &\ge \limsup_{p\to\infty} \xi_p^{1/p} e^{-\rho_p} = \limsup_{p\to\infty} \xi_p^{1/p},\end{aligned}$$

which completes the proof of the lemma. □

We proceed with the proof of the theorem.

Lemma 11.5.6. *For every $d \in \mathbb{R}^2$ and $n \in \mathbb{N}$ we have*

$$e^{-n\rho_n\|d\|} K_2^{-\|d\|} p^{-k} \Phi_n(d) \le \exp[nP_J(\langle d, \Phi\rangle)] \le C^{\|d\|}\Phi_n(d).$$

Proof of the lemma. By Lemma 11.5.4 we have

$$\Phi_{n+l}(d) \le C^{\|d\|}\Phi_n(d)\Phi_l(d),$$

and hence,

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n}\log \Phi_n(d) &= \lim_{n\to\infty} \frac{1}{n}\log(C^{\|d\|}\Phi_n(d)) \\ &= \inf_{n\in\mathbb{N}} \frac{1}{n}\log(C^{\|d\|}\Phi_n(d)).\end{aligned} \tag{11.25}$$

By Theorem 10.1.3 and (11.17), we have

$$P_J(\langle d, \Phi\rangle) = \lim_{n\to\infty} \frac{1}{n}\log \sum_{i_1\cdots i_n} \Phi_n(i_1\cdots i_n, d) = \lim_{n\to\infty} \frac{1}{n}\log \Phi_n(d),$$

and it follows from (11.25) that

$$\exp[nP_J(\langle d, \Phi\rangle)] \le C^{\|d\|}\Phi_n(d) \quad \text{for each} \quad n \in \mathbb{N}.$$

Again by Lemma 11.5.4, we have

$$\Phi_{l+n}(d) \ge \Phi_l(d)\Phi_n(d)p^{-k}e^{-n\widetilde{\rho}_n},$$

where $\widetilde{\rho}_n = \|d\|(\rho_n + \frac{1}{n}\log K_2)$, and it follows from Lemma 11.5.5 that

$$\exp P_J(\langle d, \Phi\rangle) = \lim_{n\to\infty} \Phi_n(d)^{1/n} \ge \sup_{n\in\mathbb{N}}[(\Phi_n(d)p^{-k})^{1/n}e^{-\widetilde{\rho}_n}].$$

Hence,

$$\exp[nP_J(\langle d, \Phi\rangle)] \ge \Phi_n(d)p^{-k}e^{-n\widetilde{\rho}_n} \quad \text{for each} \quad n \in \mathbb{N}.$$

This completes the proof of the lemma. □

For each $n \in \mathbb{N}$, we define a probability measure $\nu_{n,d}$ in the algebra generated by the sets $R_{i_1 \cdots i_n}$ by

$$\nu_{n,d}(R_{i_1 \cdots i_n}) = \frac{\Phi_n(i_1 \cdots i_n, d)}{\Phi_n(d)},$$

and we extend it arbitrarily to the Borel σ-algebra of J. Then there exists a subsequence $(\nu_{n_k,d})_{k \in \mathbb{N}}$ converging to a probability measure ν_d when $k \to \infty$.

In the following lemmas we establish several properties of the measures ν_d. Our first result shows that each ν_d has a weak Gibbs property.

Lemma 11.5.7. *There exists $K > 0$ such that*

$$p^{-2k}(Ke^{2n\rho_n})^{-\|d\|} \le \frac{\nu_d(R_{i_1 \cdots i_n})}{\exp[-nP_J(\langle d, \Phi \rangle)]\Phi_n(i_1 \cdots i_n, d)} \le p^k (Ke^{n\rho_n})^{\|d\|}$$

for every $d \in \mathbb{R}^2$, $n \in \mathbb{N}$, and $(i_1 \cdots i_n) \in S_n$.

Proof of the lemma. Take $q \in \mathbb{N}$ such that A^q has only integer entries, and take integers $n \in \mathbb{N}$ and $l > n + k$. Since $\nu_{l,d}$ is a measure in the algebra generated by the sets $R_{i_1 \cdots i_l}$, we have

$$\nu_{l,d}(R_{i_1 \cdots i_n}) = \sum_{j_1 \cdots j_{l-n}} \frac{\Phi_l(i_1 \cdots i_n j_1 \cdots j_{l-n}, d)}{\Phi_l(d)},$$

where the sum is taken over all $(j_1 \cdots j_{l-n})$ such that $(i_1 \cdots i_n j_1 \cdots j_{l-n}) \in S_l$. By Lemmas 11.5.3, 11.5.4, and 11.5.6, we obtain

$$\begin{aligned}
\nu_{l,d}(R_{i_1 \cdots i_n}) &\le \Phi_n(i_1 \cdots i_n, d)\Phi_{l-n}(d)\Phi_l(d)^{-1} C^{\|d\|} \\
&\le \Phi_n(i_1 \cdots i_n, d)\Phi_l(d)\Phi_n(d)^{-1} e^{n\rho_n \|d\|}\Phi_l(d)^{-1} p^k (CK_2)^{\|d\|} \\
&\le \Phi_n(i_1 \cdots i_n, d)\exp[-nP_J(\langle d, \Phi \rangle)] e^{n\rho_n \|d\|} p^k (C^2 K_2)^{\|d\|}.
\end{aligned}$$

Analogously, we also have

$$\begin{aligned}
\nu_{l,d}(R_{i_1 \cdots i_n}) &\ge \Phi_n(i_1 \cdots i_n, d)\Phi_{l-n}(d)\Phi_l(d)^{-1} e^{-n\rho_n \|d\|} p^{-k} K_2^{-\|d\|} \\
&\ge \Phi_n(i_1 \cdots i_n, d)\Phi_l(d)\Phi_n(d)^{-1}\Phi_l(d)^{-1} e^{-n\rho_n \|d\|} p^{-k} (CK_2)^{-\|d\|} \\
&\ge \Phi_n(i_1 \cdots i_n, d)\exp[-nP_J(\langle d, \Phi \rangle)] e^{-2n\rho_n \|d\|} p^{-2k} (CK_2^2)^{-\|d\|}.
\end{aligned}$$

Therefore, taking the limit of the sequence $\nu_{n_k,d}$ when $k \to \infty$ yields the desired inequalities. □

By Lemma 11.5.7 and Proposition 11.3.4 we obtain the following statement.

Lemma 11.5.8. *For every $d \in \mathbb{R}^2$, $n \in \mathbb{N}$, and $x \in R_{i_1 \cdots i_n}$ we have*

$$p^{-2k}(Ke^{3n\rho_n})^{-\|d\|} \le \frac{\nu_d(R_{i_1 \cdots i_n})}{\exp[-nP_J(\langle d, \Phi \rangle) + \langle d, \varphi_n(x) \rangle]} \le p^k (Ke^{2n\rho_n})^{\|d\|}.$$

In particular, for each $x = \chi(i_1 i_2 \cdots) \in E(\alpha)$ we have

$$P_J(\langle d, \Phi\rangle) + \lim_{n\to\infty} \frac{1}{n} \log \nu_d(R_{i_1\cdots i_n}) = \langle d, \alpha\rangle \tag{11.26}$$

(in particular, the limit in (11.26) is well defined). We emphasize that the measure ν_d may not be invariant and thus, the existence of the limit in (11.26) does not follow from the Shannon–McMillan–Breiman theorem.

For the proof of the ergodicity of ν_d we need the following statement.

Lemma 11.5.9. *There exists a constant $K_3 > 0$ such that*

$$\begin{aligned}
&\sum_{k_1\cdots k_{m-n-l}} \Phi_m(i_1 \cdots i_n k_1 \cdots k_{m-n-l} j_1 \cdots j_l, d) \\
&\geq \Phi_n(i_1 \cdots i_n, d)\Phi_l(j_1 \cdots j_l, d)\Phi_{m-n-l}(d) e^{(-n\rho_n - l\rho_l)\|d\|} p^{-2k} K_3^{-\|d\|}
\end{aligned}$$

for every $d \in \mathbb{R}^2$, $n, l \in \mathbb{N}$, $(i_1 \cdots i_n) \in S_n$, $(j_1 \cdots j_l) \in S_l$, and $m > n + l + 2k$.

Proof of the lemma. Since A^q has only positive entries, for each $\hat{L} \in S_{m-n-l-2k}$ there exist $L_1, L_2 \in S_k$, with $k = q - 1$, such that

$$(i_1 \cdots i_n L_1 \hat{L} L_2 j_1 \cdots j_l) \in S_m.$$

Arguing in a similar manner to that in (11.23) and (11.24), we obtain

$$\begin{aligned}
&\sum_{k_1\cdots k_{m-n-l}} \Phi_m(i_1 \cdots i_n k_1 \cdots k_{m-n-l} j_1 \cdots j_l, d) \\
&\geq \sum_{\hat{L}} \Phi_n(i_1 \cdots i_n, d)\Phi_l(j_1 \cdots j_l, d) e^{(-n\rho_n - l\rho_l)\|d\|} \\
&\quad \times \Phi_{m-n-l-2k}(\hat{L}, d) C^{-4\|d\|} C'^{-2k\|d\|} \\
&= \Phi_n(i_1 \cdots i_n, d)\Phi_l(j_1 \cdots j_l, d) e^{(-n\rho_n - l\rho_l)\|d\|} \Phi_{m-n-l-2k}(d) C^{-4\|d\|} C'^{-2k\|d\|}.
\end{aligned}$$

The desired inequality follows readily from Lemma 11.5.2. □

We can now establish the ergodicity of the measures ν_d.

Lemma 11.5.10. *Each measure ν_d is ergodic.*

Proof of the lemma. Given sets $R_{i_1\cdots i_n}$ and $R_{j_1\cdots j_l}$, for each $m > n + 2k$ we have

$$\nu_d(R_{i_1\cdots i_n} \cap f^{-m} R_{j_1\cdots j_l}) = \sum_{k_1\cdots k_{m-n}} \nu_d(R_{i_1\cdots i_n k_1\cdots k_{m-n} j_1\cdots j_l}).$$

By Lemmas 11.5.7, 11.5.9, and 11.5.6, we obtain

$$\begin{aligned}
&\nu_d(R_{i_1\cdots i_n} \cap f^{-m}R_{j_1\cdots j_l}) \\
&\geq \sum_{k_1\cdots k_{m-n}} \Phi_{m+l}(i_1\cdots i_n k_1\cdots k_{m-n} j_1\cdots j_l, d) \\
&\quad \times p^{-2k} \exp[-(m+l)(P_J(\langle d, \Phi\rangle) + 2\rho_{m+l}\|d\|)]K^{-\|d\|} \\
&\geq \exp[-(m+l)(P_J(\langle d, \Phi\rangle) + 2\rho_{m+l}\|d\|)] \\
&\quad \times \Phi_n(i_1\cdots i_n, d)\Phi_l(j_1\cdots j_l, d)\Phi_{m-n}(d)e^{(-n\rho_n - l\rho_l)\|d\|}p^{-4k}(KK_3)^{-\|d\|} \\
&\geq \exp[-(n+l)P_J(\langle d, \Phi\rangle) - 2(m+l)\rho_{m+l}\|d\|] \\
&\quad \times \Phi_n(i_1\cdots i_n, d)\Phi_l(j_1\cdots j_l, d)e^{(-n\rho_n - l\rho_l)\|d\|}p^{-4k}(CKK_3)^{-\|d\|}.
\end{aligned}$$

Then, it follows from Lemma 11.5.7 that

$$\begin{aligned}
&\nu_d(R_{i_1\cdots i_n} \cap f^{-m}R_{j_1\cdots j_l}) \\
&\geq \nu_d(R_{i_1\cdots i_n})\nu_d(R_{j_1\cdots j_l})e^{-2(n\rho_n + l\rho_l + (m+l)\rho_{m+l})\|d\|}p^{-6k}(CK^3K_3)^{-\|d\|}.
\end{aligned} \tag{11.27}$$

Given Borel sets $A, B \subset J$, we write them as disjoint unions up to zero measure sets with respect to the measure ν_d, that is,

$$A = \bigcup_{i=1}^{\infty} R_{a_i} \pmod 0 \quad \text{and} \quad B = \bigcup_{j=1}^{\infty} R_{b_j} \pmod 0.$$

For each $m \in \mathbb{N}$, we have

$$\begin{aligned}
\nu_d(f^{-m}A \cap B) &= \nu_d\left(\bigcup_{i=1}^{\infty} f^{-m}R_{a_i} \cap \bigcup_{j=1}^{\infty} R_{b_j}\right) \\
&= \nu_d\left(\bigcup_{i,j=1}^{\infty} f^{-m}R_{a_i} \cap R_{b_j}\right) = \sum_{i,j=1}^{\infty} \nu_d(f^{-m}R_{a_i} \cap R_{b_j}).
\end{aligned}$$

Let us assume that $\nu_d(A) > 0$ and $\nu_d(B) > 0$. Then we take finite sequences $a_i \in S_{l_i}$ and $b_j \in S_{l_j}$ with $\nu_d(R_{a_i}) > 0$ and $\nu_d(R_{b_j}) > 0$. For each integer $m > l_i + 2k$, it follows from (11.27) that

$$\nu_d(f^{-m}R_{a_i} \cap R_{b_j}) \geq \nu_d(R_{a_i})\nu_d(R_{b_j})D > 0,$$

for some constant $D = D(l_i, l_j, m) > 0$, and thus

$$\nu_d(f^{-m}A \cap B) > 0. \tag{11.28}$$

Now let A be an f-invariant set and take $B = J \setminus A$. If $0 < \nu_d(A) < 1$, then

$$0 = \nu_d(A \cap (J \setminus A)) = \nu_d(f^{-m}A \cap (J \setminus A))$$

for every $m \in \mathbb{N}$, while it follows from (11.28) that

$$\nu_d(f^{-m}A \cap (J \setminus A)) > 0.$$

This contradiction shows that either $\nu_d(A) = 0$ or $\nu_d(A) = 1$. In other words, the measure ν_d is ergodic. □

The statement in Theorem 11.5.1 follows now readily from Lemmas 11.5.7 and 11.5.10. □

We note that for an additive sequence Φ any measure ν_d satisfying (11.16) is a weak Gibbs measure in the sense of Yuri in [200] (more precisely, she allows the inequalities in (11.16) to hold almost everywhere). For an arbitrary almost additive sequence Φ it is still reasonable to continue calling each measure ν_d a *weak Gibbs measure* for Φ. We refer to Oliveira [143] and to Oliveira and Viana [144] for related results concerning the study of equilibrium measures for a class of nonuniformly expanding maps.

11.6 Multifractal analysis of Lyapunov exponents

For the class of nonconformal repellers satisfying a cone condition and having bounded distortion, we describe in this section a multifractal analysis for the entropy spectrum of the Lyapunov exponents. More precisely, we consider the sets $E(\alpha)$ in (11.3) and we describe the entropy spectrum $\alpha \mapsto h(f|E(\alpha))$ using the almost additive thermodynamic formalism.

Theorem 11.6.1 ([10]). *Let J be a repeller of a $C^{1+\varepsilon}$ map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ which is topologically mixing on J, such that:*

1. *f satisfies a cone condition on J;*
2. *f has bounded distortion on J.*

Then for each gradient $\alpha \in \mathbb{R}^2$ of the topological pressure:

1. $$h(f|E(\alpha)) = \inf_{d\in\mathbb{R}^2}[P_J(\langle d, \Phi\rangle) - \langle d, \alpha\rangle]; \tag{11.29}$$

2. *there exists a unique ergodic f-invariant probability measure $\mu_{d(\alpha)}$ in J concentrated on $E(\alpha)$ such that*
$$P_J(\langle d(\alpha), \Phi\rangle) = h_{\mu_{d(\alpha)}}(f) + \langle d(\alpha), \alpha\rangle; \tag{11.30}$$

3. *there exists a constant $K > 0$ (independent of α) such that*
$$K^{-(1+\|d(\alpha)\|)} \le \frac{\mu_{d(\alpha)}(R_{i_1\cdots i_n})}{\exp[-nP_J(\langle d(\alpha), \Phi\rangle) + \langle d(\alpha), \varphi_n(x)\rangle]} \le K^{1+\|d(\alpha)\|} \tag{11.31}$$
for every $n \in \mathbb{N}$, $(i_1 \cdots i_n) \in S_n$, and $x \in R_{i_1\cdots i_n}$.

Proof. We first establish an auxiliary statement.

Lemma 11.6.2. *There exist a constant* $\widetilde{K} > 0$ *and for each* $d \in \mathbb{R}^2$ *a unique ergodic* f*-invariant probability measure* μ_d *among the measures* ν_d*, such that*

$$\widetilde{K}^{-(1+\|d\|)} \le \frac{\mu_d(R_{i_1\cdots i_n})}{\exp[-nP_J(\langle d,\Phi\rangle) + \langle d,\varphi_n(x)\rangle]} \le \widetilde{K}^{1+\|d\|} \tag{11.32}$$

for every $n \in \mathbb{N}$*,* $(i_1 \cdots i_n) \in S_n$*, and* $x \in R_{i_1\cdots i_n}$*.*

Proof of the lemma. By Lemma 11.5.8 and the bounded distortion property of f, there exists a constant $\widetilde{K} > 0$ such that

$$\widetilde{K}^{-(1+\|d\|)} \le \frac{\nu_d(R_{i_1\cdots i_n})}{\exp[-nP_J(\langle d,\Phi\rangle) + \langle d,\varphi_n(x)\rangle]} \le \widetilde{K}^{1+\|d\|}$$

for every $d \in \mathbb{R}^2$, $n \in \mathbb{N}$, $(i_1 \cdots i_n) \in S_n$, and $x \in R_{i_1\cdots i_n}$.

Now we consider the sequence of measures

$$\left(\frac{1}{n}\sum_{l=0}^{n-1} \nu_d \circ f^{-l}\right)_{n\in\mathbb{N}}. \tag{11.33}$$

Clearly, any sublimit μ_d of this sequence is an f-invariant probability measure concentrated on J. Moreover, it satisfies (11.32). Indeed, applying successively Lemma 11.5.7, (11.21), Lemma 11.5.6, and again Lemma 11.5.7 we obtain

$$\begin{aligned}
\nu_d(f^{-l}R_{i_1\cdots i_n}) &= \sum_{j_1\cdots j_l} \nu_d(R_{j_1\cdots j_l i_1\cdots i_n}) \\
&\le c_1 \sum_{j_1\cdots j_l} \exp[-(l+n)P_J(\langle d,\Phi\rangle)]\Phi_{l+n}(j_1\cdots j_l i_1\cdots i_n, d) \\
&\le c_2 \sum_{j_1\cdots j_l} \exp[-(l+n)P_J(\langle d,\Phi\rangle)]\Phi_l(j_1\cdots j_l, d)\Phi_n(i_1\cdots i_n, d) \\
&= c_2 \exp[-(l+n)P_J(\langle d,\Phi\rangle)]\Phi_l(d)\Phi_n(i_1\cdots i_n, d) \\
&\le c_3 \exp[-nP_J(\langle d,\Phi\rangle)]\Phi_n(i_1\cdots i_n, d) \le c_4\nu_d(R_{i_1\cdots i_n}),
\end{aligned}$$

for some constants $c_1, c_2, c_3, c_4 > 0$. Similarly, applying successively Lemma 11.5.7, (11.22)–(11.23), Lemma 11.5.6, and again Lemma 11.5.7 we obtain

$$\begin{aligned}
\nu_d(f^{-l}R_{i_1\cdots i_n}) &\ge c_5 \sum_{j_1\cdots j_l} \exp[-(l+n)P_J(\langle d,\Phi\rangle)]\Phi_{l+n}(j_1\cdots j_l i_1\cdots i_n, d) \\
&\ge c_6 \sum_{j_1\cdots j_l} \exp[-(l+n)P_J(\langle d,\Phi\rangle)]\Phi_l(j_1\cdots j_l, d)\Phi_n(i_1\cdots i_n, d) \\
&\ge c_7 \exp[-nP_J(\langle d,\Phi\rangle)]\Phi_n(i_1\cdots i_n, d) \ge c_8\nu_d(R_{i_1\cdots i_n}),
\end{aligned}$$

for some constants $c_5, c_6, c_7, c_8 > 0$. Therefore,

$$c_8 \nu_d(R_{i_1\cdots i_n}) \le \frac{1}{n}\sum_{l=0}^{n-1} \nu_d(f^{-l}R_{i_1\cdots i_n}) \le c_4 \nu_d(R_{i_1\cdots i_n}) \tag{11.34}$$

for every $n \in \mathbb{N}$. This implies that any limit point μ_d of the sequence of measures in (11.33) satisfies (11.32). We could now proceed in a similar manner to that in the proof of Lemma 11.5.10 to show that μ_d is ergodic. Alternatively, by (11.34) we have

$$c_8 \nu_d(R_{i_1\cdots i_n}) \le \mu_d(R_{i_1\cdots i_n}) \le c_4 \nu_d(R_{i_1\cdots i_n}) \tag{11.35}$$

for every $n \in \mathbb{N}$ and $(i_1 \cdots i_n) \in S_n$. By Lemma 11.5.10, the measure ν_d is ergodic, and hence, by (11.35), the measure μ_d is also ergodic. The uniqueness of μ_d follows from its ergodicity together with the fact that by (11.32) any two such measures must be equivalent. □

We have the following variational principle for the topological pressure, where the measures μ_d are given by Lemma 11.6.2.

Lemma 11.6.3. *If f has bounded distortion on J, then for every $d \in \mathbb{R}^2$ we have*

$$P_J(\langle d, \Phi\rangle) = \max_{\mu\in\mathcal{M}_f}\left(h_\mu(f) + \int_J \lim_{n\to\infty}\frac{1}{n}\langle d, \varphi_n\rangle\, d\mu\right), \tag{11.36}$$

and

$$P_J(\langle d, \Phi\rangle) = h_{\mu_d}(f) + \int_J \lim_{n\to\infty}\frac{1}{n}\langle d, \varphi_n\rangle\, d\mu_d. \tag{11.37}$$

Proof of the lemma. Identity (11.36) is a particular case of Theorem 10.1.5.

To establish identity (11.37) we need the following special case of the extension of Kingman's subadditive ergodic theorem given by Derriennic in [46] (in fact, this particular statement follows from the subadditive ergodic theorem, since the sequence $\psi_n = \varphi_n + C$ is subadditive).

Lemma 11.6.4. *Let $f\colon J \to J$ be a continuous transformation of a compact metric space. If $(\varphi_n)_{n\in\mathbb{N}}$ is an almost additive sequence of continuous functions in J, then for each f-invariant probability measure μ in J, the sequence φ_n/n converges μ-almost everywhere and in $L^1(J,\mu)$.*

By Lemma 11.6.4, there is an f-invariant μ-measurable function $\overline{\varphi}$ such that $\varphi_n/n \to \overline{\varphi}$ when $n \to \infty$ μ-almost everywhere and in $L^1(J,\mu)$. Hence, it follows from Lemma 11.6.2 and the Shannon–McMillan-Breiman theorem that

$$P_J(\langle d, \Phi\rangle) = \lim_{n\to\infty} -\frac{1}{n}\log \mu_d(R_{i_1\cdots i_n}) + \lim_{n\to\infty}\frac{1}{n}\langle d, \varphi_n(x)\rangle$$

for μ_d-almost every $x \in J$ (where $x \in R_{i_1\cdots i_n}$ for each $n \in \mathbb{N}$). Integrating over J, we obtain

$$\begin{aligned} P_J(\langle d, \Phi\rangle) &= \int_J \lim_{n\to\infty} -\frac{1}{n} \log \mu_d(R_{i_1\cdots i_n})\, d\mu_d(x) + \int_J \lim_{n\to\infty} \frac{1}{n}\langle d, \varphi_n(x)\rangle\, d\mu_d(x) \\ &= h_{\mu_d}(f) + \int_J \lim_{n\to\infty} \frac{1}{n}\langle d, \varphi_n\rangle\, d\mu_d. \end{aligned}$$

This completes the proof of the lemma. □

We also establish the convexity of the function Q.

Lemma 11.6.5. *The function $Q\colon d \mapsto P_J(\langle d, \Phi\rangle)$ is convex.*

Proof of the lemma. By Hölder's inequality, given sequences $\Phi = (\varphi_n)_{n\in\mathbb{N}}$ and $\Psi = (\psi_n)_{n\in\mathbb{N}}$ of continuous functions $\varphi_n, \psi_n\colon J \to \mathbb{R}$ and $t \in (0,1)$ we have

$$\begin{aligned} &\sum_{i_1\cdots i_n} \exp\left[t \sup_{x\in R_{i_1\cdots i_n}} \varphi_n(x) + (1-t) \sup_{x\in R_{i_1\cdots i_n}} \psi_n(x)\right] \\ &\le \left(\sum_{i_1\cdots i_n} \exp \sup_{x\in R_{i_1\cdots i_n}} \varphi_n(x)\right)^t \left(\sum_{i_1\cdots i_n} \exp \sup_{x\in R_{i_1\cdots i_n}} \psi_n(x)\right)^{1-t}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\log \sum_{i_1\cdots i_n} \exp \sup_{x\in R_{i_1\cdots i_n}} (t\varphi_n(x) + (1-t)\psi_n(x)) \\ &\le t \log \sum_{i_1\cdots i_n} \exp \sup_{x\in R_{i_1\cdots i_n}} \varphi_n(x) + (1-t) \log \sum_{i_1\cdots i_n} \exp \sup_{y\in R_{i_1\cdots i_n}} \psi_n(y), \end{aligned}$$

which implies that

$$P_J(t\Phi + (1-t)\Psi) \le tP_J(\Phi) + (1-t)P_J(\Psi).$$

The convexity follows by setting $\Phi = \langle d', \varphi_n\rangle$ and $\Psi = \langle d, \varphi_n\rangle$. □

Lemma 11.6.6. *For each $\alpha = \nabla Q(d) \in \mathbb{R}^2$, the measure μ_d is concentrated on $E(\alpha)$, that is, $\mu_d(E(\alpha)) = 1$, and*

$$P_J(\langle d, \Phi\rangle) = h_{\mu_d}(f) + \langle d, \alpha\rangle. \tag{11.38}$$

Proof of the lemma. By Lemma 11.6.3, for each $v \in \mathbb{R}^2$ we have

$$P_J(\langle d+v, \Phi\rangle) \ge h_{\mu_d}(f) + \int_J \lim_{n\to\infty} \frac{1}{n}\langle d+v, \varphi_n\rangle\, d\mu_d.$$

Thus, it follows from (11.37) that

$$\begin{aligned}
&P_J(\langle d+v,\Phi\rangle)-P_J(\langle d,\Phi\rangle)\\
&\geq \int_J \lim_{n\to\infty}\frac{1}{n}\langle d+v,\varphi_n\rangle\,d\mu_d-\int_J \lim_{n\to\infty}\frac{1}{n}\langle d,\varphi_n\rangle\,d\mu_d\\
&=\left\langle v,\int_J \lim_{n\to\infty}\frac{1}{n}\varphi_n\,d\mu_d\right\rangle.
\end{aligned}$$

This shows that

$$q=\int_J \lim_{n\to\infty}\frac{\varphi_n}{n}\,d\mu_d \tag{11.39}$$

is a subgradient of Q at the point d. By Lemma 11.6.5, the function Q is convex, and since it is differentiable at d, the derivative $\alpha=\nabla Q(d)$ is the unique subgradient at this point (see for example [163]). In particular,

$$\alpha=\nabla Q(d)=q. \tag{11.40}$$

By Lemma 11.6.4, there is an f-invariant measurable function $\overline{\varphi}$ such that $\varphi_n/n\to\overline{\varphi}$ when $n\to\infty$ μ_d-almost everywhere. On the other hand, by Lemma 11.5.10, the measure μ_d is ergodic, and thus $\overline{\varphi}$ is constant μ_d-almost everywhere. Together with (11.39) and (11.40) this shows that μ_d is concentrated on the level set $E(\alpha)$. Finally, identity (11.38) follows readily from (11.37) in Lemma 11.6.3. □

We can now establish identity (11.29) in the theorem. By inequality (11.32), for each $x\in E(\alpha)$ we have

$$\lim_{n\to\infty}-\frac{1}{n}\log\mu_d(R_{i_1\cdots i_n})=P_J(\langle d,\Phi\rangle)-\langle d,\alpha\rangle,$$

where $x\in R_{i_1\cdots i_n}$ for each $n\in\mathbb{N}$. Now let $\alpha=\nabla Q(d)$ be a gradient of the topological pressure. By Lemma 11.6.6, we have $\mu_d(E(\alpha))=1$, and thus, it follows from the Shannon–McMillan–Breiman theorem that

$$h(f|E(\alpha))\geq\inf_{d\in\mathbb{R}^2}[P_J(\langle d,\Phi\rangle)-\langle d,\alpha\rangle]. \tag{11.41}$$

Now we establish the reverse inequality. We first observe that the topological entropy $h=h(f|E(\alpha))$ is the unique root of the equation $P_{E(\alpha)}(-h\Psi)=0$, where the sequence $\Psi=(\psi_n)_{n\in\mathbb{N}}$ is defined by $\psi_n=n$ for each $n\in\mathbb{N}$. Given $\varepsilon>0$ and $\tau\in\mathbb{N}$, we consider the set

$$L_{\varepsilon,\tau}=\bigl\{x\in J:|\varphi_{i,n}(x)-\alpha_i\psi_n|\leq\varepsilon n\text{ for every }n\geq\tau,\ i=1,2\bigr\}. \tag{11.42}$$

We note that $L_{\varepsilon,\tau}\subset L_{\varepsilon,\tau'}$ for $\tau\leq\tau'$, and

$$E(\alpha)\subset\bigcap_{\varepsilon>0}\bigcup_{\tau\in\mathbb{N}}L_{\varepsilon,\tau}.$$

Using the bounded distortion property, it follows from the proof of Proposition 11.3.4 that there exists $\delta > 0$ such that

$$\sup_{y \in B_n(x,\delta)} \|\varphi_n(x) - \varphi_n(y)\| \le \varepsilon n \tag{11.43}$$

for every $x \in J$. Hence, if $B_n(x,\delta) \cap L_{\varepsilon,\tau} \neq \varnothing$, then by (11.43) and (11.42) we obtain

$$|\varphi_{i,n}(y) - \alpha_i \psi_n| \le 2\varepsilon n \quad \text{for every} \quad y \in B_n(x,\delta).$$

By the second property in Theorem 4.2.2, this implies that

$$P_{L_{\varepsilon,\tau}}(-h\Psi) \le P_{L_{\varepsilon,\tau}}(\langle d, (\varphi_n - \alpha\psi_n)_{n\in\mathbb{N}}\rangle - h\Psi) + 2\varepsilon\|d\|,$$

and hence,

$$\begin{aligned} 0 &= P_{E(\alpha)}(-h\Psi) \\ &\le P_{\bigcup_{\tau\in\mathbb{N}} L_{\varepsilon,\tau}}(-h\Psi) = \sup_{\tau\in\mathbb{N}} P_{L_{\varepsilon,\tau}}(-h\Psi) \\ &\le P_J(\langle d, (\varphi_n - \alpha\psi_n)_{n\in\mathbb{N}}\rangle - h\Psi) + 2\varepsilon\|d\|. \end{aligned}$$

Since ε is arbitrary, we obtain

$$\inf_{d\in\mathbb{R}^2} P_J(\langle d, (\varphi_n - \alpha\psi_n)_{n\in\mathbb{N}}\rangle - h\Psi) = \inf_{d\in\mathbb{R}^2} [P_J(\langle d, \Phi\rangle) - \langle d, \alpha\rangle] - h \ge 0.$$

Together with (11.41) this completes the proof of identity (11.29).

The properties in (11.30) and (11.31) are simple rewritings respectively of identity (11.38) in Lemma 11.6.6, and of inequality (11.32) in Lemma 11.6.2. This completes the proof of the theorem. □

We mention some works related to Theorem 11.6.1. Feng and Lau [68] and Feng [63, 65] studied products of nonnegative matrices and their thermodynamic properties (see also Section 7.5). Barreira and Radu [18] obtained lower bounds for the dimension spectra of a class of repellers of nonconformal transformations. In a related direction, Jordan and Simon [105] established formulas for the dimension spectra of almost all self-affine maps in the plane (although their results generalize to any dimension). For repellers and hyperbolic sets of $C^{1+\varepsilon}$ conformal maps, Pesin and Weiss [156, 157] effected a multifractal analysis of the dimension spectrum. We also refer to [23, 173, 181] for other related work. In [16], Barreira, Pesin and Schmeling obtained a multifractal analysis of the entropy spectrum for repellers of $C^{1+\varepsilon}$ expanding maps that are not necessarily conformal. In [188], Takens and Verbitski effected a multifractal analysis of the entropy spectrum for expansive homeomorphisms with specification and equilibrium measures of a certain class of continuous functions (we note that these systems need not have Markov partitions). Corresponding versions for hyperbolic flows were obtained by Barreira and Saussol in [19] in the case of entropy spectra and by Pesin and Sadovskaya in [154] in the case of dimension spectra.

We also mention briefly a few directions of research concerning nonuniformly hyperbolic systems and countable topological Markov chains. In [159], Pollicott and Weiss presented a multifractal analysis of the Lyapunov exponent for the Gauss map and for the Manneville–Pomeau transformation. Related results were obtained by Yuri in [201]. In [130, 131, 132], Mauldin and Urbański developed the theory of infinite conformal iterated function systems, studying in particular the Hausdorff dimension of the limit set (see also [85]). Related results were obtained by Nakaishi in [140]. In [112], Kesseböhmer and Stratmann established a detailed multifractal analysis for Stern–Brocot intervals, continued fractions, and certain Diophantine growth rates, building on their former work [111]. In [98], Iommi obtained a detailed multifractal analysis for countable topological Markov chains, using the so-called Gurevich pressure introduced by Sarig in [169] (building on former work of Gurevich [82]). In [12], Barreira and Iommi considered the case of suspension flows over a countable topological Markov chain, building also on work of Savchenko [170] on the notion of topological entropy. In [100], Iommi and Skorulski studied the multifractal analysis of conformal measures for the exponential family $z \mapsto \lambda e^z$ with $\lambda \in (0, 1/e)$ (we note that in this setting the Julia set is not compact and that the dynamics is not Markov on the Julia set). They used a construction described by Urbański and Zdunik in [193]. For more recent work, we refer to [13, 76, 77, 99, 101, 102, 103, 104, 145].

Chapter 12

Multifractal Analysis

We consider in this chapter some further developments of the multifractal analysis of almost additive sequences. These sequences are asymptotically additive and thus the theory described in Chapter 9 also applies. Nevertheless, being a smaller class it is also possible to develop the theory further, in several directions. We also consider from the beginning the so-called u-dimension spectra of which the entropy spectra (considered in Chapter 9) and the dimension spectra are special cases. We first establish a conditional variational principle for the spectra of almost additive sequences, as an application of the almost additive thermodynamic formalism developed in Chapter 10. Essentially, this corresponds to describing the level sets of certain generalized Birkhoff averages in terms of appropriate equilibrium measures provided by the almost additive formalism. We also show that the spectra are continuous. We then give a complete description of the dimension spectra of the generalized Birkhoff averages of an almost additive sequence in a conformal hyperbolic set. We emphasize that we consider simultaneously averages into the future and into the past. More precisely, the spectra are obtained by computing the Hausdorff dimension of the level sets of the generalized Birkhoff averages both for positive and negative time. We also consider in this chapter the general case of multidimensional sequences, that is, of vectors of almost additive sequences.

12.1 Conditional variational principle

The main objective of this section is to establish a conditional variational principle for the u-dimension spectra of almost additive sequences (we recall the notion of u-dimension in Section 12.1.2). We also show that the spectra are continuous and that the corresponding irregular sets where the generalized Birkhoff averages are not defined have full dimension (even though they have zero measure with respect to any invariant measure). For simplicity of the exposition, we first describe the results in the simpler case of the entropy spectra.

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12.1.1 The case of entropy spectra

In this section we formulate our main result (Theorem 12.1.4) in the simpler case of the entropy spectra. In particular, the statement does not require the notion of u-dimension.

Let $f\colon X \to X$ be a continuous transformation of a compact metric space, and let $\Phi = (\varphi_n)_{n\in\mathbb{N}}$ be an almost additive sequence of continuous functions $\varphi_n\colon X \to \mathbb{R}$. Given $\alpha \in \mathbb{R}$, we consider the level set

$$K_\alpha = \left\{x \in X : \lim_{n\to\infty} \frac{\varphi_n(x)}{n} = \alpha\right\}.$$

The *entropy spectrum* $\mathcal{E}\colon \mathbb{R} \to \mathbb{R}$ (of the sequence Φ) is defined by

$$\mathcal{E}(\alpha) = h(f|K_\alpha),$$

where $h(f|K_\alpha)$ denotes the topological entropy of f on K_α (we recall the definition in Section 12.1.2). We also consider the function $\mathcal{P}\colon \mathcal{M}_f \to \mathbb{R}$ defined by

$$\mathcal{P}(\mu) = \lim_{n\to\infty} \frac{1}{n} \int_X \varphi_n \, d\mu$$

in the set $\mathcal{M}_f$ of the f-invariant probability measures in X.

The following result of Barreira and Doutor in [8] is a conditional variational principle for the entropy spectrum $\mathcal{E}$. It is a consequence of Theorem 12.1.4 below.

Theorem 12.1.1. *Let f be a continuous transformation of a compact metric space such that the map $\mu \mapsto h_\mu(f)$ is upper semicontinuous, and let Φ be an almost additive sequence with tempered variation and with a unique equilibrium measure. If $\alpha \notin \mathcal{P}(\mathcal{M}_f)$, then $K_\alpha = \varnothing$. Otherwise, if $\alpha \in \operatorname{int}\mathcal{P}(\mathcal{M}_f)$, then $K_\alpha \neq \varnothing$, and the following properties hold:*

1. *we have the conditional variational principle*

$$\mathcal{E}(\alpha) = \max\left\{h_\mu(f) : \mu \in \mathcal{M}_f \text{ and } \mathcal{P}(\mu) = \alpha\right\};$$

2. $\mathcal{E}(\alpha) = \min\{P_X(q\Phi) - q\alpha : q \in \mathbb{R}\}$;
3. *there is an ergodic measure $\mu_\alpha \in \mathcal{M}_f$ such that*

$$\mathcal{P}(\mu_\alpha) = \alpha, \quad \mu_\alpha(K_\alpha) = 1, \quad \text{and} \quad h_{\mu_\alpha}(f) = \mathcal{E}(\alpha).$$

In addition, the spectrum $\mathcal{E}$ is continuous in $\operatorname{int}\mathcal{P}(\mathcal{M}_f)$.

In the case of additive sequences the statement in Theorem 12.1.1 was first obtained by Barreira and Saussol in [21]. Also in the additive case, Feng, Lau and Wu [69] established a version of Theorem 12.1.1 for the dimension spectrum.

12.1.2 The notion of u-dimension

We recall in this section the notion of u-dimension introduced by Barreira and Schmeling in [23]. We note that the topological entropy is a particular case of the u-dimension.

Let again $f\colon X \to X$ be a continuous transformation of a compact metric space, and let $\mathcal{V}$ be a finite open cover of X. Let also $u\colon X \to \mathbb{R}$ be a positive continuous function. Given $Z \subset X$ and $\alpha \in \mathbb{R}$, we define

$$N(Z, \alpha, u, \mathcal{V}) = \lim_{n\to\infty} \inf_{\Gamma} \sum_{V\in\Gamma} \exp(-\alpha u(V)),$$

where $u(V)$ is given by (2.3), and where the infimum is taken over all collections $\Gamma \subset \bigcup_{k\geq n} \mathcal{W}_k(\mathcal{V})$ such that $\bigcup_{u\in\Gamma} X(V) \supset Z$. We also define

$$\dim_{u,\mathcal{V}} Z = \inf\bigl\{\alpha \in \mathbb{R}\colon N(Z, \alpha, u, \mathcal{V}) = 0\bigr\}.$$

One can show that the limit

$$\dim_u Z = \lim_{\operatorname{diam}\mathcal{V}\to 0} \dim_{u,\mathcal{V}} Z$$

exists, and we call it the *u-dimension* of the set Z (with respect to f). We note that if $u = 1$, then $\dim_u Z$ coincides with the topological entropy $h(f|Z)$ of f in the set Z.

The following result is an easy consequence of the definitions.

Proposition 12.1.2. *The u-dimension $\dim_u Z$ is the unique root of the equation $P_Z(-\alpha U) = 0$, where the sequence $U = (u_n)_{n\in\mathbb{N}}$ is defined by $u_n = \sum_{k=0}^{n-1} u \circ f^k$ for each $n \in \mathbb{N}$.*

Furthermore, given a probability measure μ in X, we define

$$\dim_{u,\mathcal{V}} \mu = \inf\bigl\{\dim_{u,\mathcal{V}} Z : \mu(Z) = 1\bigr\}.$$

One can show that the limit

$$\dim_u \mu = \lim_{\operatorname{diam}\mathcal{V}\to 0} \dim_{u,\mathcal{V}} \mu$$

exists, and we call it the *u-dimension* of the measure μ. Moreover, the *lower* and *upper u-pointwise dimensions* of μ at the point $x \in X$ are defined by

$$\underline{d}_{\mu,u}(x) = \lim_{\operatorname{diam}\mathcal{V}\to 0} \liminf_{n\to\infty} \inf_{V} -\frac{\log \mu(X(V))}{u(V)}$$

and

$$\overline{d}_{\mu,u}(x) = \lim_{\operatorname{diam}\mathcal{V}\to 0} \limsup_{n\to\infty} \sup_{V} -\frac{\log \mu(X(V))}{u(V)},$$

where the infimum and supremum are taken over all vectors $V \in \mathcal{W}_n(\mathcal{V})$ such that $x \in X(V)$. If the measure $\mu \in \mathcal{M}_f$ is ergodic, then

$$\dim_u \mu = \underline{d}_{\mu,u}(x) = \overline{d}_{\mu,u}(x) = \frac{h_\mu(f)}{\int_X u\,d\mu} \tag{12.1}$$

for μ-almost every $x \in X$ (see [23]).

12.1.3 Conditional variational principle

We present in this section a conditional variational principle for the u-dimension spectrum of an almost additive sequence. It was obtained by Barreira and Doutor in [8], and it contains as a particular case Theorem 12.1.1 for the entropy spectrum. Their approach builds on former work of Barreira, Saussol and Schmeling [22] in the case of additive sequences.

Let $\kappa \in \mathbb{N}$ and take $(A, B) \in A(X)^\kappa \times A(X)^\kappa$. We write

$$A = (\Phi_1, \dots, \Phi_\kappa) \quad \text{and} \quad B = (\Psi_1, \dots, \Psi_\kappa),$$

and also $\Phi_i = (\varphi_{i,n})_{n\in\mathbb{N}}$ and $\Psi_i = (\psi_{i,n})_{n\in\mathbb{N}}$ for $i = 1, \dots, \kappa$. We always assume that

$$\liminf_{m\to\infty} \frac{\psi_{i,m}(x)}{m} > 0 \quad \text{and} \quad \psi_{i,n}(x) > 0$$

for $i = 1, \dots, \kappa$, $x \in X$, and $n \in \mathbb{N}$. Given $\alpha = (\alpha_1, \dots, \alpha_\kappa) \in \mathbb{R}^\kappa$, we consider the level set

$$K_\alpha = \bigcap_{i=1}^{\kappa} \left\{ x \in X : \lim_{n\to\infty} \frac{\varphi_{i,n}(x)}{\psi_{i,n}(x)} = \alpha_i \right\}. \tag{12.2}$$

Definition 12.1.3. The function $\mathcal{F}_u \colon \mathbb{R}^\kappa \to \mathbb{R}$ defined by

$$\mathcal{F}_u(\alpha) = \dim_u K_\alpha$$

is called the *u-dimension spectrum of the pair* (A, B) (with respect to f).

We also consider the function $\mathcal{P} \colon \mathcal{M}_f \to \mathbb{R}^\kappa$ defined by

$$\mathcal{P}(\mu) = \lim_{n\to\infty} \left(\frac{\int_X \varphi_{1,n}\,d\mu}{\int_X \psi_{1,n}\,d\mu}, \dots, \frac{\int_X \varphi_{\kappa,n}\,d\mu}{\int_X \psi_{\kappa,n}\,d\mu} \right).$$

It follows from the continuity of the map in (10.47) that the function $\mathcal{P}$ is continuous. Since $\mathcal{M}_f$ is compact and connected, the set $\mathcal{P}(\mathcal{M}_f)$ is also compact and connected.

The following is a conditional variational principle for the spectrum $\mathcal{F}_u$. Given vectors $\alpha = (\alpha_1, \dots, \alpha_\kappa)$ and $\beta = (\beta_1, \dots, \beta_\kappa)$ in $\mathbb{R}^\kappa$ we use the notation

$$\alpha * \beta = (\alpha_1\beta_1, \dots, \alpha_\kappa\beta_\kappa) \quad \text{and} \quad \langle \alpha, \beta \rangle = \sum_{i=1}^{\kappa} \alpha_i\beta_i.$$

We also consider the sequence of functions $U = (u_n)_{n\in\mathbb{N}}$ defined by

$$u_n = \sum_{k=0}^{n-1} u \circ f^k \quad \text{for each} \quad n \in \mathbb{N}.$$

Theorem 12.1.4 (Conditional variational principle [8]). *Let $f\colon X \to X$ be a continuous transformation of a compact metric space such that the map $\mu \mapsto h_\mu(f)$ is upper semicontinuous, and let us assume that*

$$\operatorname{span}\{\Phi_1, \Psi_1, \dots, \Phi_\kappa, \Psi_\kappa, U\} \subset E(X).$$

If $\alpha \notin \mathcal{P}(\mathcal{M}_f)$, then $K_\alpha = \varnothing$. Otherwise, if $\alpha \in \operatorname{int}\mathcal{P}(\mathcal{M}_f)$, then $K_\alpha \neq \varnothing$, and the following properties hold:

1. *we have the conditional variational principle*

$$\mathcal{F}_u(\alpha) = \max\left\{ \frac{h_\mu(f)}{\int_X u\, d\mu} : \mu \in \mathcal{M}_f \text{ and } \mathcal{P}(\mu) = \alpha \right\}; \tag{12.3}$$

2. $\mathcal{F}_u(\alpha) = \min\{T_u(\alpha, q) : q \in \mathbb{R}^\kappa\}$, *where $T_u(\alpha, q)$ is the unique real number satisfying*

$$P_X(\langle q, A - \alpha * B\rangle - T_u(\alpha, q)U) = 0;$$

3. *there is an ergodic measure $\mu_\alpha \in \mathcal{M}_f$ such that $\mathcal{P}(\mu_\alpha) = \alpha$, $\mu_\alpha(K_\alpha) = 1$, and*

$$\dim_u \mu_\alpha = \frac{h_{\mu_\alpha}(f)}{\int_X u\, d\mu_\alpha} = \mathcal{F}_u(\alpha). \tag{12.4}$$

In addition, the spectrum $\mathcal{F}_u$ is continuous in $\operatorname{int}\mathcal{P}(\mathcal{M}_f)$.

Proof. We first establish some auxiliary results.

Lemma 12.1.5. *If $\alpha \in \mathcal{P}(\mathcal{M}_f)$, then*

$$\inf_{q\in\mathbb{R}^\kappa} P_X(\langle q, A - \alpha * B\rangle - \mathcal{F}_u(\alpha)U) \geq 0.$$

Proof of the lemma. Given $\delta > 0$ and $\tau \in \mathbb{N}$, we consider the set

$$L_{\delta,\tau} = \big\{x \in X : \|A_n(x) - \alpha * B_n(x)\| < \delta n \text{ for every } n \geq \tau\big\},$$

where

$$A_n = (\varphi_{1,n}, \dots, \varphi_{\kappa,n}) \quad \text{and} \quad B_n = (\psi_{1,n}, \dots, \psi_{\kappa,n}),$$

and using the norm $\|\alpha\| = |\alpha_1| + \cdots + |\alpha_\kappa|$ in $\mathbb{R}^\kappa$. For each $x \in K_\alpha$ and $i = 1, \dots, \kappa$, we have

$$\lim_{n\to\infty} \frac{\varphi_{i,n}(x)}{\psi_{i,n}(x)} = \alpha_i.$$

Since $\psi_{i,n} > 0$, for each $\delta > 0$ there exists $\tau \in \mathbb{N}$ (independent of i) such that

$$\left|\frac{\varphi_{i,n}(x)}{\psi_{i,n}(x)} - \alpha_i\right| < \frac{\delta}{\kappa \max_i(\|\psi_{i,1}\|_\infty + C)}$$

for every $n > \tau$. Using (10.46) we thus obtain

$$\begin{aligned}\|A_n(x) - \alpha * B_n(x)\| &= \sum_{i=1}^{\kappa} |\varphi_{i,n}(x) - \alpha_i \psi_{i,n}(x)| \\ &< \frac{\delta}{\kappa \max_i(\|\psi_{i,1}\|_\infty + C)} \sum_{i=1}^{\kappa} \psi_{i,n}(x) < \delta n.\end{aligned}$$

This shows that $x \in L_{\delta,\tau}$, and hence,

$$K_\alpha \subset \bigcap_{\delta>0} \bigcup_{\tau \in \mathbb{N}} L_{\delta,\tau}. \tag{12.5}$$

On the other hand, since the sequence Φ_i has tempered variation, given $\delta > 0$ we have

$$\limsup_{n\to\infty} \frac{\gamma_n(\Phi_i, \mathcal{V})}{n} < \frac{\delta}{\kappa}$$

for any finite open cover $\mathcal{V}$ of X with sufficiently small diameter. Therefore,

$$|\varphi_{i,n}(x) - \varphi_{i,n}(y)| < \delta n/\kappa$$

for every $V \in \mathcal{W}_n(\mathcal{V})$ and $x, y \in X(V)$, and all sufficiently large $n \in \mathbb{N}$. Now let

$$A(V) = (\Phi_1(V), \dots, \Phi_\kappa(V)),$$

where $\Phi_i(V)$ is defined as in (2.3). For each $y \in X(V)$, we have

$$\begin{aligned}\|A(V) - A_n(y)\| &= \sum_{i=1}^{\kappa} |\Phi_i(V) - \varphi_{i,n}(y)| \\ &\le \sum_{i=1}^{\kappa} \sup_{x \in X(V)} |\varphi_{i,n}(x) - \varphi_{i,n}(y)| \le \delta n,\end{aligned}$$

and similarly,

$$\|B(V) - B_n(y)\| \le \delta n,$$

for every $V \in \mathcal{W}_n(\mathcal{V})$ and $y \in X(V)$.

Given $q \in \mathbb{R}^\kappa$, $V \in \mathcal{W}_n(\mathcal{V})$ such that $X(V) \cap L_{\delta,\tau} \ne \varnothing$, and $y \in X(V) \cap L_{\delta,\tau}$, we have

$$\begin{aligned}-\langle q, A - \alpha * B\rangle(V) &\le |\langle q, A(V) - \alpha * B(V)\rangle| \\ &\le \|q\| \cdot \|A(V) - \alpha * B(V)\| \\ &\le \|q\| \cdot \|A(V) - A_n(y)\| \\ &\quad + \|q\| \cdot \|\alpha * B_n(y) - \alpha * B(V)\| \\ &\quad + \|q\| \cdot \|A_n(y) - \alpha * B_n(y)\| \\ &\le \|q\|(\delta n + \|\alpha\|\delta n + \delta n) = (2 + \|\alpha\|)\|q\|\delta n.\end{aligned}$$

Therefore, setting $T = -\mathcal{F}_u(\alpha)U$ we obtain

$$\begin{aligned}\exp(T(V) - \beta n) &= \exp\big([T + \langle q, A - \alpha * B\rangle](V) - \langle q, A - \alpha * B\rangle(V) - \beta n\big)\\ &\le \exp\big([\langle q, A - \alpha * B\rangle + T](V) - [\beta - (2 + \|\alpha\|)\|q\|\delta]n\big)\end{aligned}$$

for every $\beta \in \mathbb{R}$. Now let us take $k \ge \tau$ and $\Gamma \subset \bigcup_{n\ge k} \mathcal{W}_n(\mathcal{V})$ such that $L_{\delta,\tau} \subset \bigcup_{V\in\Gamma} X(V)$. Without loss of generality we assume that there is no vector $V \in \Gamma$ such that $X(V) \cap L_{\delta,\tau} = \varnothing$. We obtain

$$\begin{aligned}&\sum_{V\in\Gamma} \exp(T(V) - \beta m(V))\\ &\le \sum_{V\in\Gamma} \exp\big([\langle q, A - \alpha * B\rangle + T](V) - [\beta - (2 + \|\alpha\|)\|q\|\delta]m(V)\big),\end{aligned}$$

and hence,

$$M_{L_{\delta,\tau}}(\beta, T, \mathcal{V}) \le M_{L_{\delta,\tau}}\big(\beta - (2 + \|\alpha\|)\|q\|\delta, \langle q, A - \alpha * B\rangle + T, \mathcal{V}\big).$$

Since

$$\begin{aligned}&\inf\big\{\beta : M_{L_{\delta,\tau}}\big(\beta - (2 + \|\alpha\|)\|q\|\delta, \langle q, A - \alpha * B\rangle + T, \mathcal{V}\big) = 0\big\}\\ &= \inf\big\{\gamma : M_{L_{\delta,\tau}}\big(\gamma, \langle q, A - \alpha * B\rangle + T, \mathcal{V}\big) = 0\big\} + (2 + \|\alpha\|)\|q\|\delta,\end{aligned}$$

letting $\operatorname{diam}\mathcal{V} \to 0$ we obtain

$$P_{L_{\delta,\tau}}(T) \le P_{L_{\delta,\tau}}(\langle q, A - \alpha * B\rangle + T) + (2 + \|\alpha\|)\delta\|q\|$$

for every $\delta > 0$ and $q \in \mathbb{R}^\kappa$. By Proposition 12.1.2, we have $P_{K_\alpha}(T) = 0$, and in view of (12.5) it follows from Theorem 4.2.1 that

$$\begin{aligned}0 &= P_{K_\alpha}(T)\\ &\le P_{\bigcup_{\tau\in\mathbb{N}} L_{\delta,\tau}}(T) = \sup_{\tau\in\mathbb{N}} P_{L_{\delta,\tau}}(T)\\ &\le \sup_{\tau\in\mathbb{N}} P_{L_{\delta,\tau}}(\langle q, A - \alpha * B\rangle + T) + (2 + \|\alpha\|)\delta\|q\|\\ &\le P_X(\langle q, A - \alpha * B\rangle + T) + (2 + \|\alpha\|)\delta\|q\|\end{aligned}$$

for every $\delta > 0$ and $q \in \mathbb{R}^\kappa$. The statement in the lemma follows now from the arbitrariness of δ. ☐

Lemma 12.1.6. *If* $\alpha \in \operatorname{int}\mathcal{P}(\mathcal{M}_f)$, *then*

$$\min_{q\in\mathbb{R}^\kappa} P_X(\langle q, A - \alpha * B\rangle - \mathcal{F}_u(\alpha)U) = 0,$$

and there exists an ergodic equilibrium measure $\mu_\alpha \in \mathcal{M}_f$ *such that* $\mathcal{P}(\mu_\alpha) = \alpha$, $\mu_\alpha(K_\alpha) = 1$, *and*

$$\dim_u\mu_\alpha = \frac{h_{\mu_\alpha}(f)}{\int_X u\, d\mu_\alpha} = \mathcal{F}_u(\alpha).$$

Proof of the lemma. Let $r > 0$ be the distance of α to the set $\mathbb{R}^d \setminus \mathcal{P}(\mathcal{M}_f)$. Given $\beta = (\beta_1, \ldots, \beta_\kappa) \in \mathbb{R}^\kappa$ with $\beta_i = \alpha_i + r \operatorname{sgn} q_i/(2\kappa)$ for $i = 1, \ldots, \kappa$, we have

$$\|\beta - \alpha\| = \sum_{i=1}^{\kappa} |\beta_i - \alpha_i| = \sum_{i=1}^{\kappa} \left| \frac{1}{2\kappa} r \operatorname{sgn} q_i \right| = \frac{r}{2} < r,$$

and hence $\beta \in \mathcal{P}(\mathcal{M}_f)$. Therefore, there exists a measure $\mu \in \mathcal{M}_f$ such that

$$\lim_{n\to\infty} \frac{1}{n} \int_X A_n \, d\mu = \lim_{n\to\infty} \frac{1}{n} \int_X \beta * B_n \, d\mu,$$

Now we define

$$F(q) = P_X\big(\langle q, A - \alpha * B\rangle - \mathcal{F}_u(\alpha)U\big) \tag{12.6}$$

for each $q \in \mathbb{R}^\kappa$. By the variational principle in Theorem 10.3.1, we have

$$\begin{aligned} F(q) &\geq h_\mu(f) + \lim_{n\to\infty} \frac{1}{n} \int_X [\langle q, A_n - \alpha * B_n\rangle - \mathcal{F}_u(\alpha) u_n] \, d\mu \\ &= h_\mu(f) + \lim_{n\to\infty} \frac{1}{n} \left\langle q, \int_X (A_n - \alpha * B_n) \, d\mu \right\rangle - \mathcal{F}_u(\alpha) \int_X u \, d\mu. \end{aligned} \tag{12.7}$$

On the other hand, since

$$\begin{aligned} \left\langle q, \int_X (\beta - \alpha) * B_n \, d\mu \right\rangle &= \sum_{i=1}^{\kappa} q_i \int_X (\beta_i - \alpha_i)\psi_{i,n} \, d\mu \\ &= \sum_{i=1}^{\kappa} \frac{1}{2\kappa} r q_i \operatorname{sgn} q_i \int_X \psi_{i,n} \, d\mu \\ &= \frac{1}{2\kappa} r \sum_{i=1}^{\kappa} |q_i| \int_X \psi_{i,n} \, d\mu, \end{aligned}$$

we have

$$\begin{aligned} \frac{1}{n} \left\langle q, \int_X (A_n - \alpha * B_n) \, d\mu \right\rangle =& \frac{1}{2\kappa} r \sum_{i=1}^{\kappa} |q_i| \frac{1}{n} \int_X \psi_{i,n} \, d\mu \\ &+ \frac{1}{n} \left\langle q, \int_X (A_n - \beta * B_n) \, d\mu \right\rangle. \end{aligned}$$

Taking the limit when $n \to \infty$, we obtain

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} \left\langle q, \int_X (A_n - \alpha * B_n) \, d\mu \right\rangle &= \frac{1}{2\kappa} r \sum_{i=1}^{\kappa} |q_i| \lim_{n\to\infty} \frac{1}{n} \int_X \psi_{i,n} \, d\mu \\ &\geq \frac{1}{2\kappa} r \|q\| \min_i \lim_{n\to\infty} \frac{1}{n} \int_X \psi_{i,n} \, d\mu. \end{aligned} \tag{12.8}$$

Since $h_\mu(f) \geq 0$, it follows from (12.7) and (12.8) that

$$F(q) \geq \frac{1}{2\kappa} r\|q\| \min_i \lim_{n\to\infty} \frac{1}{n} \int_X \psi_{i,n}\, d\mu - \mathcal{F}_u(\alpha) \int_X u\, d\mu. \tag{12.9}$$

We note that the right-hand side of (12.9) takes arbitrarily large values for $\|q\|$ sufficiently large. Thus, there exists $R \in \mathbb{R}$ such that $F(q) \geq F(0)$ for every $q \in \mathbb{R}^\kappa$ with $\|q\| \geq R$.

By Theorem 10.4.1, the function F in (12.6) is of class C^1, and thus it attains a minimum at some point $q = q(\alpha)$ with $\|q(\alpha)\| \leq R$. In particular, we have $d_{q(\alpha)}F = 0$. Now let μ_α be the equilibrium measure of the sequence of functions

$$\langle q(\alpha), A - \alpha * B\rangle - \mathcal{F}_u(\alpha)U.$$

By Theorem 10.4.1, we have

$$\lim_{n\to\infty} \frac{1}{n} \int_X (A_n - \alpha * B_n)\, d\mu_\alpha = d_{q(\alpha)}F = 0, \tag{12.10}$$

and thus $\mathcal{P}(\mu_\alpha) = \alpha$. Moreover,

$$F(q(\alpha)) = h_{\mu_\alpha}(f) - \mathcal{F}_u(\alpha) \int_X u\, d\mu_\alpha.$$

By Lemma 12.1.5, we have $F(q(\alpha)) \geq 0$, and hence,

$$\mathcal{F}_u(\alpha) \leq \frac{h_{\mu_\alpha}(f)}{\int_X u\, d\mu_\alpha},$$

Again by Theorem 10.4.1, the measure μ_α is ergodic, and thus,

$$\dim_u \mu_\alpha = \frac{h_{\mu_\alpha}(\Phi)}{\int_X u\, d\mu_\alpha} \geq \mathcal{F}_u(\alpha).$$

On other hand, by (12.10) we have

$$\lim_{n\to\infty} \frac{1}{n} \int_X A_n\, d\mu_\alpha = \lim_{n\to\infty} \frac{1}{n} \int_X \alpha * B_n\, d\mu_\alpha,$$

and thus $\mu_\alpha(K_\alpha) = 1$. Hence,

$$\mathcal{F}_u(\alpha) = \dim_u K_\alpha \geq \dim_u \mu_\alpha,$$

and we conclude that $\dim_u \mu_\alpha = \mathcal{F}_u(\alpha)$. Therefore, it follows from (12.1) that

$$\begin{aligned}\min_{q\in\mathbb{R}^\kappa} P_\Phi(\langle q, A - \alpha * B\rangle - \mathcal{F}_u(\alpha)U) &= F(q(\alpha)) \\ &= h_{\mu_\alpha}(\Phi) - \mathcal{F}_u(\alpha) \int_X u\, d\mu_\alpha \\ &= h_{\mu_\alpha}(\Phi) - \frac{h_{\mu_\alpha}(\Phi)}{\int_X u\, d\mu_\alpha} \int_X u\, d\mu_\alpha = 0.\end{aligned}$$

This completes the proof of the lemma. ∎

We proceed with the proof of the theorem. Take $\alpha \in \mathbb{R}^\kappa$ such that $K_\alpha \neq \varnothing$. For each $x \in K_\alpha$, we have

$$\lim_{n\to\infty} \frac{\varphi_{i,n}(x)}{\psi_{i,n}(x)} = \alpha_i \quad \text{for} \quad i = 1, \dots, \kappa. \tag{12.11}$$

Now we consider the sequence $(\mu_n)_{n\in\mathbb{N}}$ of probability measures in X defined by

$$\mu_n = \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x)}.$$

The set $V(x)$ of all sublimits of this sequence is nonempty, and clearly $V(x) \subset \mathcal{M}_f$.

Let us take $\delta > 0$, $i \in \{1, \dots, \kappa\}$, and a measure $\mu \in V(x)$. Proceeding in a similar manner to that in the proof of Theorem 10.1.5 (see (10.14)), we find that there is an increasing sequence $(n_k)_{k\in\mathbb{N}}$ of natural numbers such that

$$\lim_{k\to\infty} \frac{\varphi_{i,n_k}(x)}{n_k} = \lim_{n\to\infty} \frac{1}{n} \int_X \varphi_{i,n}\, d\mu, \tag{12.12}$$

and

$$\lim_{k\to\infty} \frac{\psi_{i,n_k}(x)}{n_k} = \lim_{n\to\infty} \frac{1}{n} \int_X \psi_{i,n}\, d\mu. \tag{12.13}$$

It follows from (12.11) together with (12.12) and (12.13) that

$$\lim_{n\to\infty} \frac{\int_X \varphi_{i,n}\, d\mu}{\int_X \psi_{i,n}\, d\mu} = \frac{\lim_{n\to\infty} \frac{1}{n} \int_X \varphi_{i,n}\, d\mu}{\lim_{n\to\infty} \frac{1}{n} \int_X \psi_{i,n}\, d\mu} = \alpha_i.$$

Therefore,

$$\mathcal{P}(\mu) = \lim_{n\to\infty} \left(\frac{\int_X \varphi_{1,n}\, d\mu}{\int_X \psi_{1,n}\, d\mu}, \dots, \frac{\int_X \varphi_{\kappa,n}\, d\mu}{\int_X \psi_{\kappa,n}\, d\mu} \right) = \alpha,$$

and we conclude that $\alpha \in \mathcal{P}(\mathcal{M}_f)$.

Now let us take $\alpha \in \operatorname{int} \mathcal{P}(\mathcal{M}_f)$ and $\mu \in \mathcal{M}_f$ with $\mathcal{P}(\mu) = \alpha$. By the upper semicontinuity of the function

$$\mu \mapsto \frac{h_\mu(f)}{\int_X u\, d\mu},$$

the existence of the maximum in (12.3) follows readily from the compactness of $\mathcal{M}_f$ together with the continuity of $\mathcal{P}$. To show that the maximum coincides with $\mathcal{F}_u(\alpha)$, we notice that by Lemma 12.1.6 and Theorem 10.3.1,

$$\begin{aligned} 0 &= \inf_{q\in\mathbb{R}^\kappa} P_X\big(\langle q, A - \alpha * B\rangle - \mathcal{F}_u(\alpha)U\big) \\ &\geq \inf_{q\in\mathbb{R}^\kappa} \left\{ h_\mu(f) + \lim_{n\to\infty} \frac{1}{n} \int_X [\langle q, A_n - \alpha * B_n\rangle - \mathcal{F}_u(\alpha)u_n]\, d\mu \right\}. \end{aligned}$$

Since $\mathcal{P}(\mu) = \alpha$, we have

$$\lim_{n\to\infty} \frac{1}{n} \int_X \langle q, A_n - \alpha * B_n \rangle \, d\mu = 0,$$

and thus,

$$0 \ge \inf_{q\in\mathbb{R}^\kappa} \left\{ h_\mu(f) - \mathcal{F}_u(\alpha) \int_X u \, d\mu \right\} = h_\mu(f) - \mathcal{F}_u(\alpha) \int_X u \, d\mu.$$

Therefore,

$$\frac{h_\mu(f)}{\int_X u \, d\mu} \le \mathcal{F}_u(\alpha). \tag{12.14}$$

On other hand, by Lemma 12.1.6 there exists an ergodic equilibrium measure μ_α such that $\mathcal{P}(\mu_\alpha) = \alpha$, $\mu_\alpha(K_\alpha) = 1$, and

$$\dim_u \mu_\alpha = \frac{h_{\mu_\alpha}(f)}{\int_X u \, d\mu_\alpha} = \mathcal{F}_u(\alpha).$$

Together with (12.14) this establishes identities (12.3) and (12.4) (in particular, the set K_α is nonempty).

Now let us take $q(\alpha) \in \mathbb{R}^\kappa$ such that

$$P_X\big(\langle q(\alpha), A - \alpha * B \rangle - \mathcal{F}_u(\alpha) U\big) = 0.$$

Proceeding as in the proof of Lemma 12.1.6, we obtain

$$\mathcal{F}_u(\alpha) = T_u(\alpha, q(\alpha)) \ge \inf\{T_u(\alpha, q) : q \in \mathbb{R}^\kappa\}.$$

On other hand, again by Lemma 12.1.6 and the definition of T_u, we have

$$P_X\big(\langle q, A - \alpha * B \rangle - \mathcal{F}_u(\alpha) U\big) \ge 0 = P_X\big(\langle q, A - \alpha * B \rangle - T_u(\alpha, q) U\big)$$

for every $q \in \mathbb{R}^\kappa$. Therefore,

$$\mathcal{F}_u(\alpha) \le \inf\{T_u(\alpha, q) : q \in \mathbb{R}^\kappa\}.$$

This establishes property 2 in the theorem.

Now we establish the continuity of the spectrum. Take $\alpha \in \operatorname{int} \mathcal{P}(\mathcal{M}_f)$ and let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence in $\operatorname{int} \mathcal{P}(\mathcal{M}_f)$ converging to α. For each $n \in \mathbb{N}$, let us take $q_n \in \mathbb{R}^\kappa$ such that $\mathcal{F}_u(\alpha_n) = T_u(\alpha_n, q_n)$. We also consider $q(\alpha) \in \mathbb{R}^\kappa$ such that $\mathcal{F}_u(\alpha) = T_u(\alpha, q(\alpha))$. The existence of these vectors is guaranteed by the second property in the theorem. By the same property, we have

$$T_u(\alpha_n, q_n) = \min_{q\in\mathbb{R}^\kappa} T_u(\alpha_n, q) \le T_u(\alpha_n, q(\alpha)),$$

and hence,

$$\limsup_{n\to\infty} \mathcal{F}_u(\alpha_n) \le \mathcal{F}_u(\alpha).$$

On other hand, since $\mathcal{F}_u(\alpha)$ is a minimum, it follows from the continuity of the function T_u (which is of class C^1) that

$$\liminf_{n\to\infty} \mathcal{F}_u(\alpha_n) = \liminf_{n\to\infty} T_u(\alpha_n, q_n) \ge \mathcal{F}_u(\alpha).$$

This shows that $\mathcal{F}_u(\alpha_n) \to \mathcal{F}_u(\alpha)$ when $n \to \infty$, and the proof of the theorem is complete. □

12.1.4 Irregular sets

This section is dedicated to a brief study of the irregular sets, that is, the sets for which the limits in (12.2) do not exist. For simplicity of the exposition, we only consider the particular case of topological Markov chains.

Given almost additive sequences $\Phi_1, \Psi_1, \ldots, \Phi_\kappa, \Psi_\kappa$ as in Section 12.1.3 with $X = \Sigma_A^+$, we consider the *irregular set*

$$I = \bigcap_{i=1}^{\kappa} \left\{ x \in \Sigma_A^+ : \liminf_{n\to\infty} \frac{\varphi_{i,n}(x)}{\psi_{i,n}(x)} < \limsup_{n\to\infty} \frac{\varphi_{i,n}(x)}{\psi_{i,n}(x)} \right\},$$

and we denote by m_u the equilibrium measure of the function u, when it is unique.

Theorem 12.1.7 ([8]). *For a topologically mixing topological Markov chain $\sigma|\Sigma_A^+$, if*

$$\operatorname{span}\left\{\Phi_1, \Psi_1, \ldots, \Phi_\kappa, \Psi_\kappa, U\right\} \subset E(\Sigma_A^+),$$

and $\mathcal{P}(m_u) \in \operatorname{int} \mathcal{P}(\mathcal{M}_\sigma)$, then

$$\dim_u I = \dim_u \Sigma_A^+.$$

Proof. Take $\varepsilon > 0$ and $\alpha_0 \in \operatorname{int} \mathcal{P}(\mathcal{M}_\sigma)$. For $i = 1, \ldots, \kappa$, let us take $\alpha_i \in \operatorname{int} \mathcal{P}(\mathcal{M}_\sigma)$ such that $\alpha_{0,i} \ne \alpha_{i,i}$ and

$$\mathcal{F}_u(\alpha_i) > \mathcal{F}_u(\alpha_0) - \varepsilon. \tag{12.15}$$

This is always possible in view of the continuity of the spectrum $\mathcal{F}_u$ (see Theorem 12.1.4). Moreover, by the third property in Theorem 12.1.4, for $i = 0, \ldots, \kappa$ there is an ergodic measure $\mu_i \in \mathcal{M}_\sigma$ with $\mathcal{P}(\mu_i) = \alpha_i$ such that

$$\mu_i(K_{\alpha_i}) = 1 \quad \text{and} \quad \dim_u \mu_i = \mathcal{F}_u(\alpha_i).$$

This allows us to apply Theorem 7.2 in [23] to conclude that

$$\dim_u I \ge \min\{\dim_u \mu_0, \dim_u \mu_1, \ldots, \dim_u \mu_d\}.$$

Together with (12.15) this implies that $\dim_u I \geq \mathcal{F}_u(\alpha_0) - \varepsilon$, and it follows from the arbitrariness of ε and α_0 that

$$\dim_u I \geq \sup_{\alpha \in \operatorname{int} \mathcal{P}(\mathcal{M}_\sigma)} \mathcal{F}_u(\alpha). \tag{12.16}$$

On the other hand, by Proposition 12.1.2, we have

$$0 = P_{\Sigma_A^+}\big(-(\dim_u \Sigma_A^+)U\big) = h_{m_u}(\sigma) - \dim_u \Sigma_A^+ \int_{\Sigma_A^+} u \, dm_u,$$

and thus, by (12.1),

$$\dim_u m_u = \frac{h_{m_u}(\sigma)}{\int_{\Sigma_A^+} u \, dm_u} = \dim_u \Sigma_A^+.$$

Moreover, since $m_u(K_{\mathcal{P}(m_u)}) = 1$, we have $\dim_u m_u \leq \dim_u K_{\mathcal{P}(m_u)}$, and hence,

$$\mathcal{F}_u(\mathcal{P}(m_u)) \geq \dim_u m_u = \dim_u \Sigma_A^+.$$

Since $\mathcal{P}(m_u) \in \operatorname{int} \mathcal{P}(\mathcal{M}_\sigma)$, together with (12.16) this yields the statement in the theorem. □

We refer to the book [7] for a detailed study of the u-dimension of irregular sets using the classical thermodynamic formalism, and in particular for a detailed proof of Theorem 7.2 in [23] (which is used in the proof of Theorem 12.1.7). We emphasize that the u-dimension of an irregular set is unrelated to any particular thermodynamic formalism. For this reason, we do not strive to present the most general results in this section. Nevertheless, since in the present context we are considering irregular sets defined by almost additive sequences it is quite natural and in fact useful to use the almost additive version of the thermodynamic formalism in the proof.

12.2 Dimension Spectra

The main objective of this section is to give a complete description of the dimension spectra of the generalized Birkhoff averages of an almost additive sequence in a conformal hyperbolic set. We consider *simultaneously* Birkhoff averages into the future and into the past. This causes the results to be impossible to obtain simply by applying the results in Section 12.1. The main difficulty is that although the local product structure given by the intersection of local stable and unstable manifolds is a Lipschitz homeomorphism with Lipschitz inverse, the level sets of the generalized Birkhoff averages are never compact. This causes their box dimension to be strictly larger than their Hausdorff dimension, and thus a priori the Hausdorff dimension of a product of level sets need not be the sum of the dimensions of the level sets. This forces us to construct explicitly noninvariant measures concentrated on each product of level sets, with the appropriate pointwise dimension.

12.2.1 Dimension along the stable and unstable directions

We establish in this section formulas for the dimension of the level sets of the generalized Birkhoff averages contained in any stable or unstable manifold. These formulas were obtained by Barreira and Doutor in [9]. Their approach builds on former work of Barreira and Valls [24] in the case of additive sequences.

Let $f\colon M \to M$ be a $C^{1+\varepsilon}$ diffeomorphism, for some $\varepsilon \in (0,1]$, and let $\Lambda \subset M$ be a hyperbolic set for f. We always assume in this section that:

1. Λ is locally maximal (that is, there exists an open neighborhood U of Λ such that $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n U$);
2. f is conformal on Λ (see Definition 6.2.7);
3. f is topologically mixing on Λ.

Now let t_s and t_u be the unique real numbers satisfying (6.10). By Theorem 6.2.8, we have

$$\dim_H(\Lambda \cap V^s(x)) = t_s \quad \text{and} \quad \dim_H(\Lambda \cap V^u(x)) = t_u, \tag{12.17}$$

and

$$\begin{aligned} \dim_H(\Lambda \cap V^s(x)) &= \underline{\dim}_B(\Lambda \cap V^s(x)) = \overline{\dim}_B(\Lambda \cap V^s(x)), \\ \dim_H(\Lambda \cap V^u(x)) &= \underline{\dim}_B(\Lambda \cap V^u(x)) = \overline{\dim}_B(\Lambda \cap V^u(x)) \end{aligned} \tag{12.18}$$

for every $x \in \Lambda$. Moreover, by Theorem 6.2.9,

$$\begin{aligned} \dim_H \Lambda &= \dim_H[(\Lambda \cap V^s(x)) \times (\Lambda \cap V^u(x))] \\ &= \dim_H(\Lambda \cap V^s(x)) + \dim_H(\Lambda \cap V^u(x)) = t_s + t_u. \end{aligned} \tag{12.19}$$

Using the notion of bounded variation (see Definition 10.1.8, where we have fixed some Markov partition), we denote by $L^+(\Lambda)$ the family of all almost additive sequences of continuous functions with respect to f also having bounded variation with respect to f. Similarly, we denote by $L^-(\Lambda)$ the family of all almost additive sequences of continuous functions with respect to f^{-1} also having bounded variation with respect to f^{-1}.

Given $\kappa \in \mathbb{N}$, we consider the spaces

$$H^\pm(\Lambda) \subset L^\pm(\Lambda)^\kappa \times L^\pm(\Lambda)^\kappa$$

of pairs of sequences

$$A^\pm = (\Phi_1^\pm, \ldots, \Phi_\kappa^\pm) \quad \text{and} \quad B^\pm = (\Psi_1^\pm, \ldots, \Psi_\kappa^\pm),$$

where $\Phi_i^\pm = (\varphi_{i,n}^\pm)_{n\in\mathbb{N}}$ and $\Psi_i^\pm = (\psi_{i,n}^\pm)_{n\in\mathbb{N}}$, such that

$$\liminf_{m\to\infty} \frac{\psi_{i,m}^\pm(x)}{m} > 0 \quad \text{and} \quad \psi_{i,n}^\pm(x) > 0 \tag{12.20}$$

for every $n \in \mathbb{N}$, $x \in \Lambda$, and $i = 1, \ldots, \kappa$. Given

$$(\Phi^\pm, \Psi^\pm) \in H^\pm(\Lambda) \quad \text{and} \quad \alpha = (\alpha_1, \ldots, \alpha_\kappa) \in \mathbb{R}^\kappa,$$

we define

$$K_\alpha^\pm = \bigcap_{i=1}^{\kappa} \left\{ x \in \Lambda : \lim_{n\to\infty} \frac{\varphi_{i,n}^\pm(x)}{\psi_{i,n}^\pm(x)} = \alpha_i \right\}.$$

Theorem 12.2.1 ([9]). *Let Λ be a locally maximal hyperbolic set of a $C^{1+\varepsilon}$ diffeomorphism f which is conformal and topologically mixing on Λ. For each $\alpha \in \mathbb{R}^\kappa$ and $x^\pm \in K_\alpha^\pm$, we have*

$$\dim_H K_\alpha^+ = \dim_H (K_\alpha^+ \cap V^u(x^+)) + t_s,$$

and

$$\dim_H K_\alpha^- = \dim_H (K_\alpha^- \cap V^s(x^-)) + t_u.$$

Proof. Let us write $\Phi = \Phi_i^+$ and $\Psi = \Psi_i^+$. Since Φ and Ψ have bounded variation, given $\delta > 0$ there exists $\varepsilon > 0$ such that if $\mathcal{U}$ is a finite open cover of Λ with diameter at most ε, then there exists $k = k(\mathcal{U}) \in \mathbb{N}$ such that

$$|\varphi_n(x) - \varphi_n(y)| < \delta n \quad \text{and} \quad |\psi_n(x) - \psi_n(y)| < \delta n$$

for every $n \geq k$, $U \in \mathcal{W}_n(\mathcal{U})$, and $x, y \in \Lambda(U)$. Now let τ be the Lebesgue number of $\mathcal{U}$. Given $x \in \Lambda$ and $y \in V^s(x)$, there exists $p \in \mathbb{N}$ such that if $m > p$, then $d(f^m(x), f^m(y)) \leq \tau$. In particular, for each $m > p$ there exists $U_m \in \mathcal{U}$ containing $f^m(x)$ and $f^m(y)$. Given $n \geq k$, we consider the vector

$$U = (U_{p+1}, \ldots, U_{p+n}) \in \mathcal{W}_n(\mathcal{U}).$$

Since $f^{p+1}(y), f^{p+1}(x) \in \Lambda(U)$, we have

$$|\varphi_n(f^{p+1}(x)) - \varphi_n(f^{p+1}(y))| < \delta n,$$

and

$$|\psi_n(f^{p+1}(x)) - \psi_n(f^{p+1}(y))| < \delta n.$$

Since Φ is almost additive, for each $n > k + p + 1$ the absolute value of

$$\varphi_n(y) - \varphi_n(x) - \varphi_{p+1}(y) - \varphi_{n-p-1}(f^{p+1}(y)) + \varphi_{p+1}(x) + \varphi_{n-p-1}(f^{p+1}(x))$$

is at most $2C$, and thus,

$$\begin{aligned} |\varphi_n(y) - \varphi_n(x)| &\leq 2C + |\varphi_{p+1}(y) - \varphi_{p+1}(x)| \\ &\quad + |\varphi_{n-p-1}(f^{p+1}(y)) - \varphi_{n-p-1}(f^{p+1}(x))| \\ &\leq 2C + 2(p+1)(C + \|\varphi_1\|_\infty) + \delta(n - p - 1), \end{aligned}$$

with an analogous estimate for $|\psi_n(y) - \psi_n(x)|$.

On the other hand, by (12.20) there exists $\sigma(x) > 0$ such that $\psi_n(x)/n \geq \sigma(x)$ for every $n \in \mathbb{N}$. Therefore,

$$\begin{aligned}
&\left|\frac{\varphi_n(y)}{\psi_n(y)} - \frac{\varphi_n(x)}{\psi_n(x)}\right| \\
&= \left|\frac{\varphi_n(y)}{\psi_n(y)} - \frac{\varphi_n(x)}{\psi_n(y)} + \frac{\varphi_n(x)}{\psi_n(y)} - \frac{\varphi_n(x)}{\psi_n(x)}\right| \\
&\leq \frac{|\varphi_n(y) - \varphi_n(x)|}{\psi_n(y)} + \frac{|\varphi_n(x)| \cdot |\psi_n(x) - \psi_n(y)|}{\psi_n(y)\psi_n(x)} \\
&\leq \frac{2C + 2(p+1)(C + \|\varphi_1\|_\infty) + \delta(n-p-1)}{\sigma(y)n} \\
&\quad + \frac{n(C + \|\varphi_1\|_\infty)\bigl[2C + 2(p+1)(C + \|\psi_1\|_\infty) + \delta(n-p-1)\bigr]}{\sigma(y)\sigma(x)n^2},
\end{aligned}$$

and hence,

$$\limsup_{n\to\infty} \left|\frac{\varphi_n(y)}{\psi_n(y)} - \frac{\varphi_n(x)}{\psi_n(x)}\right| \leq \frac{\delta}{\sigma(y)} + \frac{(C + \|\varphi_1\|_\infty)\delta}{\sigma(y)\sigma(x)}.$$

Since δ is arbitrarily small, we conclude that the limit

$$\lim_{n\to\infty} \frac{\varphi_n(x)}{\psi_n(x)}$$

exists if and only if the limit

$$\lim_{n\to\infty} \frac{\varphi_n(y)}{\psi_n(y)}$$

exists, in which case the two limits are equal. This shows that $\Lambda \cap V^s(x) \subset K_\alpha^+$ for every $x \in K_\alpha^+$.

Since the map f is conformal on Λ, the open neighborhood

$$\Lambda \cap \bigcup_{y \in K_\alpha^+ \cap V^u(x)} V^s(y) \subset K_\alpha^+$$

of a point $x \in K_\alpha^+$ (with respect to the induced topology on Λ) is taken onto the product

$$(K_\alpha^+ \cap V^u(x)) \times (\Lambda \cap V^s(x))$$

by a Lipschitz map with Lipschitz inverse. Therefore,

$$\begin{aligned}
\dim_H K_\alpha^+ &= \dim_H (K_\alpha^+ \cap V^u(x)) + \dim_H (\Lambda \cap V^s(x)) \\
&= \dim_H (K_\alpha^+ \cap V^u(x)) + t_s,
\end{aligned}$$

in view of (12.17) and (12.18). The corresponding identity for the sets K_α^- can be obtained in a similar manner. This completes the proof of the theorem. $\square$

12.2.2 Construction of a noninvariant measure

Here and in the following section we establish some auxiliary results that shall be used in the study of the dimension spectra in Theorem 12.2.6.

We consider a Markov partition of Λ, and the associated two-sided shift $\sigma|\Sigma_A$ with transition matrix A. We denote by Σ_A^+ and Σ_A^- respectively the sets of right-sided and left-sided infinite sequences obtained from Σ_A, and we consider the one-sided shifts $\sigma^+ : \Sigma_A^+ \to \Sigma_A^+$ and $\sigma^- : \Sigma_A^- \to \Sigma_A^-$ defined by

$$\sigma^+(i_0 i_1 \cdots) = (i_1 i_2 \cdots) \quad \text{and} \quad \sigma^-(\cdots i_{-1} i_0) = (\cdots i_{-2} i_{-1}).$$

Let also $\pi^+ : \Sigma_A \to \Sigma_A^+$ and $\pi^- : \Sigma_A \to \Sigma_A^-$ be the projections defined by

$$\pi^+(\cdots i_{-1} i_0 i_1 \cdots) = (i_0 i_1 \cdots) \quad \text{and} \quad \pi^-(\cdots i_{-1} i_0 i_1 \cdots) = (\cdots i_{-1} i_0).$$

Let $\chi : \Sigma_A \to \Lambda$ be the coding map obtained from the Markov partition. Given $x \in \Lambda$, take $\omega \in \Sigma_A$ such that $\chi(\omega) = x$. Let also $R(x)$ be a rectangle of the Markov partition that contains x. For each $\omega' \in \Sigma_A$, we have

$$\chi(\omega') \in V^u(x) \cap R(x) \quad \text{whenever} \quad \pi^-(\omega') = \pi^-(\omega),$$

and

$$\chi(\omega') \in V^s(x) \cap R(x) \quad \text{whenever} \quad \pi^+(\omega') = \pi^+(\omega).$$

Hence, if $\omega = (\cdots i_{-1} i_0 i_1 \cdots)$, then the set $V^u(x) \cap R(x)$ can be identified via the map χ with the cylinder set

$$C_{i_0}^+ = \left\{ (j_0 j_1 \cdots) \in \Sigma_A^+ : j_0 = i_0 \right\} \subset \Sigma_A^+,$$

and the set $V^s(x) \cap R(x)$ can be identified via the map χ with the cylinder set

$$C_{i_0}^- = \left\{ (\cdots j_{-1} j_0) \in \Sigma_A^- : j_0 = i_0 \right\} \subset \Sigma_A^-.$$

Lemma 12.2.2. *If $\Phi^+ \in L^+(\Lambda)$, then there exists a sequence Φ^u of continuous functions $\varphi_n^u : \Sigma_A^+ \to \mathbb{R}$ and $\gamma > 0$ such that:*

1. *for every $n \in \mathbb{N}$ and $\omega \in \Sigma_A$, we have*

$$|(\varphi_n^+ \circ \chi)(\omega) - (\varphi_n^u \circ \pi^+)(\omega)| < \gamma;$$

2. *Φ^u is almost additive and has bounded variation with respect to σ^+;*
3. *$\Phi^u \circ \pi^+$ and $\Phi^+ \circ \chi$ are almost additive and have bounded variation with respect to σ;*
4.

$$P_\Lambda(\Phi^+) = P_{\Sigma_A^+}(\Phi^u) \quad \text{and} \quad P_{\Sigma_A}(\Phi^+ \circ \chi) = P_{\Sigma_A}(\Phi^u \circ \pi^+);$$

5. $\Phi^+ \circ \chi$ *and* $\Phi^u \circ \pi^+$ *have the same equilibrium measure;*
6. *the limit*

$$\lim_{n\to\infty} \frac{\varphi_n^u(\pi^+(\omega))}{\psi_n^u(\pi^+(\omega))}$$

exists if and only if the limit

$$\lim_{n\to\infty} \frac{\varphi_n^+(\chi(\omega))}{\psi_n^+(\chi(\omega))}$$

exists, in which case the two limits are equal.

Proof. The existence of the sequence Φ^u follows essentially from a construction described by Bowen in [39, Lemma 1.6], with the necessary modifications due to the almost additivity of the sequences. Namely, let p be the number of elements of the Markov partition. For each $i = 1, \ldots, p$, we take a sequence $(\omega_{ij}) \in \Sigma_A$ such that $\omega_{i0} = i$, and we consider the function $r\colon \Sigma_A \to \Sigma_A$ defined by

$$r(\omega) = r(\cdots \omega_{-2}\omega_{-1}\omega_0\omega_1\omega_2 \cdots) = (\cdots \omega_{i,-2}\omega_{i,-1} i \omega_1\omega_2 \cdots)$$

when $\omega_0 = i$. Now let us take $n \in \mathbb{N}$ and $\omega \in C^+_{i_0\cdots i_{n-1}}$, where

$$C^+_{i_0\cdots i_{n-1}} = \bigcap_{l=0}^{n-1} (\sigma^+)^{-l} C^+_{i_l}.$$

Since $\chi(\omega)$ and $\chi(r(\omega))$ are both in $\bigcap_{l=0}^{n-1} f^{-l} R_{i_l}$, and Φ^+ has bounded variation, we have

$$|\varphi_n^+(\chi(\omega)) - \varphi_n^+(\chi(r(\omega)))| \le \gamma$$

for some constant $\gamma > 0$ (independent of ω). On the other hand,

$$\varphi_n^+(\chi(r(\cdots \omega_{-1}\omega_0\omega_1 \cdots))) = \varphi_n^+(\chi(r(\cdots \omega'_{-1}\omega'_0\omega'_1 \cdots)))$$

whenever $\omega_i = \omega'_i$ for every $i \ge 0$, and thus for each $n \in \mathbb{N}$ there exists a function $\varphi_n^u\colon \Sigma_A^+ \to \mathbb{R}$ such that

$$\varphi_n^+(\chi(r(\omega))) = \varphi_n^u(\pi^+(\omega))$$

for every $\omega \in \Sigma_A$. Then $\Phi^u = (\varphi_n^u)_{n\in\mathbb{N}}$ is the desired sequence.

For the second and third statements, let us take $\omega^+ \in \Sigma_A^+$ and $\omega \in \Sigma_A$ such that $\pi^+(\omega) = \omega^+$. We have

$$\begin{aligned}
&|\varphi_{n+m}^u(\omega^+) - \varphi_n^u(\omega^+) - \varphi_m^u((\sigma^+)^n(\omega^+))| \\
&\le |\varphi_{n+m}^u(\pi^+(\omega)) - \varphi_{n+m}^+(\chi(\omega))| + |\varphi_n^+(\chi(\omega)) - \varphi_n^u(\pi^+(\omega))| \\
&\quad + |\varphi_m^+(\chi((\sigma^+)^n(\omega))) - \varphi_m^u(\pi^+((\sigma^+)^n(\omega)))| \\
&\quad + |\varphi_{n+m}^+(\chi(\omega)) - \varphi_n^+(\chi(\omega)) - \varphi_m^+(f^n(\chi(\omega)))| \le 3\gamma + C
\end{aligned}$$

for every $n, m \in \mathbb{N}$, where C is the constant in Definition 10.1.1. On the other hand, for each $n \in \mathbb{N}$ and $\omega^+, \tilde{\omega}^+ \in \Sigma_A^+$ with $\omega^+, \tilde{\omega}^+ \in C^+_{i_0\cdots i_{n-1}}$, we have

$$\begin{aligned}|\varphi_n^u(\omega^+) - \varphi_n^u(\tilde{\omega}^+)| &\le |\varphi_n^u(\pi^+\omega) - \varphi_n^+(\chi(\omega))| + |\varphi_n^+(\chi(\tilde{\omega})) - \varphi_n^u(\pi^+\tilde{\omega})| \\ &\quad + |\varphi_n^+(\chi(\omega)) - \varphi_n^+(\chi(\tilde{\omega}))| \\ &\le 2C + \gamma,\end{aligned}$$

since $\chi(\omega)$ and $\chi(\tilde{\omega})$ are both in $\bigcap_{l=0}^{n-1} f^{-l} R_{i_l}$, and Φ^+ has bounded variation.

For statement 4, we note that by Theorem 10.1.3 one has

$$P_{\Sigma_A^+}(\Phi^u) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{(i_0\cdots i_{n-1})\in S_n} \exp \varphi_n^u(\omega^+_{i_0\cdots i_{n-1}}), \tag{12.21}$$

where $\omega^+_{i_0\cdots i_{n-1}}$ is an arbitrary element of $C^+_{i_0\cdots i_{n-1}}$, and also

$$P_\Lambda(\Phi^+) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{(i_0\cdots i_{n-1})\in S_n} \exp \varphi_n^+(\chi(\omega_{i_0\cdots i_{n-1}})), \tag{12.22}$$

where $\omega_{i_0\cdots i_{n-1}} \in \Sigma_A$ is such that $\pi^+(\omega_{i_0\cdots i_{n-1}}) = \omega^+_{i_0\cdots i_{n-1}}$. Since

$$-\gamma + \varphi_n^+(\chi(\omega_{i_0\cdots i_{n-1}})) \le \varphi_n^u(\omega^+_{i_0\cdots i_{n-1}}) \le \gamma + \varphi_n^+(\chi(\omega_{i_0\cdots i_{n-1}}))$$

for every $\omega^+_{i_0\cdots i_{n-1}} \in \Sigma_A^+$, the right-hand sides of (12.22) and (12.21) are equal.

For statement 5, in view of Theorem 10.2.1 it is sufficient to observe that by statement 1, for every σ-invariant probability measure μ in Σ_A we have

$$\lim_{n\to\infty} \frac{1}{n} \int_{\Sigma_A} (\varphi_n^+ \circ \chi)\, d\mu = \lim_{n\to\infty} \frac{1}{n} \int_{\Sigma_A} (\varphi_n^u \circ \pi^+)\, d\mu.$$

Indeed, denoting by μ the equilibrium measure of $\Phi^+ \circ \chi$, we obtain

$$\begin{aligned}P_{\Sigma_A^+}(\Phi^u \circ \pi^+) &= P_{\Sigma_A^+}(\Phi^+ \circ \chi) \\ &= h_\mu(\Phi) + \lim_{n\to\infty} \frac{1}{n} \int_{\Sigma_A} (\varphi_n^+ \circ \chi)\, d\mu \\ &= h_\mu(\Phi) + \lim_{n\to\infty} \frac{1}{n} \int_{\Sigma_A} (\varphi_n^u \circ \pi^+)\, d\mu,\end{aligned}$$

and μ is also the equilibrium measure of $\Phi^u \circ \pi^+$.

The last statement follows readily from the inequalities

$$\frac{-\gamma + \varphi_n^+(\chi(\omega))}{\gamma + \psi_n^+(\chi(\omega))} \le \frac{\varphi_n^u(\pi^+(\omega))}{\psi_n^u(\pi^+(\omega))} \le \frac{\gamma + \varphi_n^+(\chi(\omega))}{-\gamma + \psi_n^+(\chi(\omega))},$$

which hold for any sufficiently large n, as a consequence of statement 1 together with the fact that the sequence $n \mapsto \psi_n^+(\chi(\omega))$ is unbounded. □

There is an analogous version of Lemma 12.2.2 for sequences in $L^-(\Lambda)$. Now let

$$A^u = (\Phi_1^u, \dots, \Phi_\kappa^u), \quad B^u = (\Psi_1^u, \dots, \Psi_\kappa^u),$$

and

$$A^s = (\Phi_1^s, \dots, \Phi_\kappa^s), \quad B^s = (\Psi_1^s, \dots, \Psi_\kappa^s).$$

We also write

$$A_n^u = (\varphi_{1,n}^u, \dots, \varphi_{\kappa,n}^u) \quad \text{and} \quad B_n^u = (\psi_{1,n}^u, \dots, \psi_{\kappa,n}^u),$$

and

$$A_n^s = (\varphi_{1,n}^s, \dots, \varphi_{\kappa,n}^s) \quad \text{and} \quad B_n^s = (\psi_{1,n}^s, \dots, \psi_{\kappa,n}^s).$$

Given $\alpha, \beta, q^\pm \in \mathbb{R}^\kappa$, we consider the almost additive sequences

$$U = \langle q^+, \Phi^u - \alpha * \Psi^u \rangle - d^+ \sum_{k=0}^{n-1} d^u \circ \sigma^+,$$

and

$$S = \langle q^-, \Phi^s - \beta * \Psi^s \rangle - d^- \sum_{k=0}^{n-1} d^s \circ \sigma^-,$$

where

$$d^+ = \dim_H K_\alpha^+ - t_s \quad \text{and} \quad d^- = \dim_H K_\alpha^- - t_u, \tag{12.23}$$

and where

$$d^u \colon \Sigma_A^+ \to \mathbb{R} \quad \text{and} \quad d^s \colon \Sigma_A^- \to \mathbb{R}$$

are Hölder continuous functions such that $d^u \circ \pi^+$ and $d^s \circ \pi^-$ are cohomologous respectively to

$$\log \|df|E^u\| \circ \chi \quad \text{and} \quad \log \|df^{-1}|E^s\| \circ \chi$$

(the existence of the functions d^u and d^s can be established as in the proof of Lemma 12.2.2). We note that the sequences U and S have bounded variation (respectively with respect to σ^+ and σ^-).

Since f and thus also f^{-1} are topologically mixing, it follows from Theorem 10.3.2 that U has a unique equilibrium measure μ^u in Σ_A^+ (with respect to σ^+), and that S has a unique equilibrium measure μ^s in Σ_A^- (with respect to σ^-). We also consider the transformations $\mathcal{P}^\pm \colon \mathcal{M}_f(\Lambda) \to \mathbb{R}^\kappa$ defined by

$$\mathcal{P}^\pm(\mu) = \lim_{n\to\infty} \left(\frac{\int_\Lambda \varphi_{1,n}^\pm \, d\mu}{\int_\Lambda \psi_{1,n}^\pm \, d\mu}, \dots, \frac{\int_\Lambda \varphi_{\kappa,n}^\pm \, d\mu}{\int_\Lambda \psi_{\kappa,n}^\pm \, d\mu} \right).$$

The following is an immediate consequence of Theorem 12.1.4.

Lemma 12.2.3. *For each $\alpha \in \operatorname{int} \mathcal{P}^+(\mathcal{M}_f(\Lambda))$ and $\beta \in \operatorname{int} \mathcal{P}^-(\mathcal{M}_f(\Lambda))$, there exist $q^+, q^- \in \mathbb{R}^\kappa$ such that*

$$P_{\sigma^+}(U) = P_{\sigma^-}(S) = 0,$$

with the corresponding measures μ^u and μ^s satisfying

$$\lim_{n\to\infty} \frac{1}{n} \int_{\Sigma_A^+} A_n^u \, d\mu^u = \alpha * \lim_{n\to\infty} \frac{1}{n} \int_{\Sigma_A^+} B_n^u \, d\mu^u,$$

and

$$\lim_{n\to\infty} \frac{1}{n} \int_{\Sigma_A^-} A_n^u \, d\mu^s = \beta * \lim_{n\to\infty} \frac{1}{n} \int_{\Sigma_A^-} B_n^s \, d\mu^s.$$

We define measures ν^u and ν^s in the rectangle $R(x) \subset \Lambda$ by

$$\nu^u = \mu^u \circ \pi^+ \circ \chi^{-1} \quad \text{and} \quad \nu^s = \mu^s \circ \pi^- \circ \chi^{-1},$$

using the vectors $q^\pm$ in Lemma 12.2.3. We also define a measure ν in $R(x)$ by

$$\nu = \nu^u \times \nu^s. \tag{12.24}$$

We note that μ^u and μ^s are Gibbs measures (see Theorem 10.1.9), and hence,

$$\nu(R(x)) = \mu^u(C_{i_0}^+)\mu^s(C_{i_0}^-) > 0.$$

12.2.3 Estimates for the pointwise dimension

We obtain in this section lower and upper bounds for the pointwise dimension of the measure ν in (12.24).

Lemma 12.2.4. *For ν-almost every $x \in \Lambda$ we have*

$$\liminf_{r\to 0} \frac{\log \nu(B(x,r))}{\log r} \geq \dim_H K_\alpha^+ + \dim_H K_\beta^- - \dim_H \Lambda.$$

Proof. By the variational principle in Theorem 10.1.5 applied to the sequences U and S, it follows from Lemma 12.2.3 that

$$\frac{h_{\mu^u}(\sigma^+)}{\int_{\Sigma_A^+} d^u \, d\mu^u} = d^+ \quad \text{and} \quad \frac{h_{\mu^s}(\sigma^-)}{\int_{\Sigma_A^-} d^s \, d\mu^s} = d^-. \tag{12.25}$$

By the Shannon–McMillan–Breiman theorem and Birkhoff's ergodic theorem, it follows from (12.25) that for every $\varepsilon > 0$, μ^u-almost every $\omega^+ \in C_{i_0}^+$, and μ^s-almost every $\omega^- \in C_{i_0}^-$, there exists $s(\omega) \in \mathbb{N}$ such that for each $n, m > s(\omega)$ we have

$$d^+ - \varepsilon < -\frac{\log \mu^u(C_{i_0\cdots i_n}^+)}{\sum_{k=0}^n d^u((\sigma^+)^k(\omega^+))} < d^+ + \varepsilon,$$

and

$$d^- - \varepsilon < -\frac{\log \mu^s(C^-_{i_{-m}\cdots i_0})}{\sum_{k=0}^m d^s((\sigma^-)^k(\omega^-))} < d^- + \varepsilon.$$

For each sufficiently small $r > 0$, let $n = n(\omega, r)$ and $m = m(\omega, r)$ be the unique positive integers such that

$$-\sum_{k=0}^{n} d^u((\sigma^+)^k(\omega^+)) > \log r, \quad -\sum_{k=0}^{n+1} d^u((\sigma^+)^k(\omega^+)) \le \log r, \tag{12.26}$$

and

$$-\sum_{k=0}^{m} d^s((\sigma^-)^k(\omega^-)) > \log r, \quad -\sum_{k=0}^{m+1} d^s((\sigma^-)^k(\omega^-)) \le \log r. \tag{12.27}$$

Now we recall that a measure μ in a set X is said to be *weakly diametrically regular* in a subset $Z \subset X$ if there is $\eta > 1$ such that for μ-almost every $x \in Z$ and every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\mu(B(x, \eta r)) \le \mu(B(x, r))r^{-\varepsilon} \tag{12.28}$$

for $r < \delta$. If μ is weakly diametrically regular in Z, then for every $\eta > 1$, μ-almost every $x \in Z$, and every $\varepsilon > 0$, there exists $\delta = \delta(x, \varepsilon) > 0$ such that (12.28) holds for $r < \delta$. Barreira and Saussol showed in [20, Lemma 1] that any probability Borel measure in $\mathbb{R}^n$ is weakly diametrically regular.

Since the map f is conformal on Λ, there exists $\rho > 1$ (independent of r and $x = \chi(\omega)$) such that

$$B(y, r/\rho) \cap \Lambda \subset \chi(C_{i_{-m}\cdots i_n}) \subset B(x, \rho r) \tag{12.29}$$

for some point $y \in \chi(C_{i_{-m}\cdots i_n})$, where $\omega = (\cdots i_{-1} i_0 i_1 \cdots)$. We take $\eta = 2\rho$ and $\delta = \delta(\eta, y, \varepsilon) > 0$ so that (12.28) holds for $r < \delta$. Therefore,

$$\begin{aligned}
\nu(B(x,r)) &\le \nu(B(y, 2\rho\frac{r}{\rho})) \le \nu(B(y, r/\rho)) \left(\frac{r}{\rho}\right)^{-\varepsilon} \\
&\le \nu(\chi(C_{i_{-m}\cdots i_n})) \left(\frac{r}{\rho}\right)^{-\varepsilon} \\
&= \mu^u(C^+_{i_0\cdots i_n})\mu^s(C^-_{i_{-m}\cdots i_0}) \left(\frac{r}{\rho}\right)^{-\varepsilon} \\
&\le \exp\left[(-d^+ + \varepsilon)\sum_{k=0}^{n} d^u((\sigma^+)^k(\omega^+))\right] \\
&\quad \times \exp\left[(-d^- + \varepsilon)\sum_{k=0}^{m} d^s((\sigma^-)^k(\omega^-))\right] \left(\frac{r}{\rho}\right)^{-\varepsilon},
\end{aligned}$$

and hence,

$$\nu(B(x,r)) \leq \exp[(\log r + \|d^u\|_\infty)(d^+ - \varepsilon)] \times \exp[(\log r + \|d^s\|_\infty)(d^- - \varepsilon)] \left(\frac{r}{\rho}\right)^{-\varepsilon}.$$

This implies that

$$\liminf_{r\to 0} \frac{\log \nu(B(x,r))}{\log r} \geq d^+ + d^- - 2\varepsilon,$$

for ν-almost every $x \in \Lambda$. On the other hand, since the product structure is locally bi-Lipschitz, we have

$$\begin{aligned}\dim_H\Lambda &= \dim_H[(\Lambda \cap V^s(x)) \times (\Lambda \cap V^u(x))] \\ &= \dim_H(\Lambda \cap V^s(x)) + \dim_H(\Lambda \cap V^s(x)) = t_s + t_u,\end{aligned} \tag{12.30}$$

using (12.19) in the second identity. Therefore, by (12.23),

$$\liminf_{r\to 0} \frac{\log \nu(B(x,r))}{\log r} \geq \dim_H K^+_\alpha + \dim_H K^-_\beta - \dim_H\Lambda - 2\varepsilon,$$

and the desired result follows from the arbitrariness of ε. □

Lemma 12.2.5. *For every $x \in K^+_\alpha \cap K^-_\beta$ we have*

$$\limsup_{r\to 0} \frac{\log \nu(B(x,r))}{\log r} \leq \dim_H K^+_\alpha + \dim_H K^-_\beta - \dim_H\Lambda.$$

Proof. Take $x \in K^+_\alpha \cap K^-_\beta$ and $\omega \in \Sigma_A$ such that $\chi(\omega) = x$. We also consider the projections $\omega^\pm = \pi^\pm(\omega)$. By statement 6 in Lemma 12.2.2, we have

$$\lim_{n\to\infty} \frac{\varphi^u_{i,n}(\omega^+)}{\psi^u_{i,n}(\omega^+)} = \alpha_i \quad \text{and} \quad \lim_{n\to\infty} \frac{\varphi^s_{i,n}(\omega^-)}{\psi^s_{i,n}(\omega^-)} = \beta_i$$

for $i = 1, \ldots, \kappa$. Therefore, given $\varepsilon > 0$ there exists $r(\omega) \in \mathbb{N}$ such that

$$\left|\frac{\varphi^u_{i,n}(\omega^+)}{\psi^u_{i,n}(\omega^+)} - \alpha_i\right| \leq \varepsilon$$

for every $n > r(\omega)$. Hence,

$$|\varphi^u_{i,n}(\omega^+) - \alpha_i\psi^u_{i,n}(\omega^+)| \leq \varepsilon|\psi^u_{i,n}(\omega^+)| \leq \varepsilon n p_u,$$

where C is the constant in Definition 10.1.1 (without loss of generality we can take the same constant for all sequences Ψ^u_i), and where

$$p_u = C + \max_{i\in\{1,\ldots,\kappa\}} \|\psi^u_{i,1}\|_\infty.$$

Denoting by u_n the elements of the sequence U, we obtain

$$\begin{aligned} u_n(\omega^+) &\geq -\sum_{i=1}^{\kappa} |q_i^+| \cdot |\varphi_{i,n}^u(\omega^+) - \alpha_i \psi_{i,n}^u(\omega^+)| - d^+ \sum_{j=0}^{n} d^u((\sigma^+)^j(\omega^+)) \\ &\geq -n\varepsilon \|q^+\| p_u - d^+ \sum_{j=0}^{n} d^u((\sigma^+)^j(\omega^+)). \end{aligned}$$

By Lemma 12.2.3, we have $P_{\sigma^+}(U) = 0$, and since μ^u is a Gibbs measure, there exists $D > 0$ such that for every $C_{i_0}^+$ and $n \in \mathbb{N}$ we have

$$D^{-1} < \frac{\mu^u(C_{i_0 \cdots i_n}^+)}{\exp(U_n(\omega^+))} < D.$$

Hence,

$$\mu^u(C_{i_0 \cdots i_n}^+) > D^{-1} \exp\left[-d^+ \sum_{j=0}^{n} d^u((\sigma^+)^j(\omega^+)) - n\varepsilon \|q^+\| p_u \right]. \tag{12.31}$$

Similarly, we have

$$\mu^s(C_{i_{-m} \cdots i_0}^-) > D^{-1} \exp\left[-d^- \sum_{j=0}^{m} d^s((\sigma^-)^j(\omega^-)) - m\varepsilon \|q^-\| p_s \right], \tag{12.32}$$

eventually with a larger D, where

$$p_s = C + \max_{i \in \{1,\ldots,\kappa\}} \|\psi_{i,1}^s\|_\infty.$$

Since Λ is a hyperbolic set, for any sufficiently small $r > 0$ we have $n(\omega, r) > r(\omega)$ and $m(\omega, r) > r(\omega)$ (see (12.26) and (12.27)). As in (12.29), there exists $\rho > 0$ (independent of $x = \chi(\omega)$ and r) such that

$$B(x, \rho r) \supset \chi(C_{i_{-m} \cdots i_n}),$$

with $n = n(\omega, r)$ and $m = m(\omega, r)$. By (12.31) and (12.32), we obtain

$$\begin{aligned} \nu(B(x, \rho r)) &\geq \nu(\chi(C_{i_{-m} \cdots i_n})) \\ &= \mu^u(C_{i_0 \cdots i_n}^+) \mu^s(C_{i_{-m} \cdots i_0}^-) \\ &\geq D^{-2} r^{d^+ + d^-} \exp(-n\varepsilon \|q^+\| p_s) \exp(-m\varepsilon \|q^-\| p_u) \end{aligned}$$

for every sufficiently small $r > 0$. By (12.26) and (12.27), we have

$$-n \inf d^u > \log r \quad \text{and} \quad -m \inf d^s > \log r,$$

and hence, for every $x \in K_\alpha^+ \cap K_\beta^-$ we obtain

$$\limsup_{r\to 0} \frac{\log \nu(B(x,r))}{\log r} \le d^+ + d^- + \varepsilon \frac{\|q^+\|p_u}{\inf d^u} + \varepsilon \frac{\|q^-\|p_s}{\inf d^s}.$$

Since ε is arbitrary, we conclude that

$$\limsup_{r\to 0} \frac{\log \nu(B(x,r))}{\log r} \le d^+ + d^-,$$

which together with (12.23) and (12.30) yields the desired result. □

12.2.4 Dimension spectra

We consider in this section the *dimension spectrum* $\mathcal{D} : \mathbb{R}^\kappa \times \mathbb{R}^\kappa \to \mathbb{R}$ defined by

$$\mathcal{D}(\alpha, \beta) = \dim_H(K_\alpha^+ \cap K_\beta^-).$$

The following is a conditional variational principle for the dimension spectrum. We denote by $\mathcal{M}_f(\Lambda)$ the set of all f-invariant probability measures in Λ.

Theorem 12.2.6 ([9]). *Let Λ be a locally maximal hyperbolic set of a $C^{1+\varepsilon}$ diffeomorphism f which is conformal and topologically mixing on Λ. Given $(A^\pm, B^\pm) \in H^\pm(\Lambda)$, if $\alpha \in \operatorname{int} \mathcal{P}^+(\mathcal{M}_f(\Lambda))$ and $\beta \in \operatorname{int} \mathcal{P}^-(\mathcal{M}_f(\Lambda))$, then*

$$\begin{aligned}
\mathcal{D}(\alpha, \beta) &= \dim_H K_\alpha^+ + \dim_H K_\beta^- - \dim_H \Lambda \\
&= \max\left\{ \frac{h_\mu(f)}{-\int_\Lambda \log \|df|E^s\| \, d\mu} : \mu \in \mathcal{M}_f(\Lambda) \text{ and } \mathcal{P}^+(\mu) = \alpha \right\} \\
&\quad + \max\left\{ \frac{h_\mu(f)}{\int_\Lambda \log \|df|E^u\| \, d\mu} : \mu \in \mathcal{M}_f(\Lambda) \text{ and } \mathcal{P}^-(\mu) = \beta \right\}.
\end{aligned}$$

Moreover, the spectrum $\mathcal{D}$ is analytic in $\operatorname{int} \mathcal{P}^+(\mathcal{M}_f(\Lambda)) \times \operatorname{int} \mathcal{P}^-(\mathcal{M}_f(\Lambda))$.

Proof. Given $\alpha \in \operatorname{int} \mathcal{P}^+(\mathcal{M}_f(\Lambda))$ and $\beta \in \operatorname{int} \mathcal{P}^-(\mathcal{M}_f(\Lambda))$, let ν be the measure constructed in (12.24). It follows from Theorem 1.4.4 and Lemma 12.2.4 that

$$\dim_H \nu = \dim_H K_\alpha^+ + \dim_H K_\beta^- - \dim_H \Lambda,$$

where $\dim_H \nu$ is the Hausdorff dimension of the measure ν (see Definition 1.4.2). By construction, we have $\nu(K_\alpha^+ \cap K_\beta^-) = 1$, and hence,

$$\dim_H(K_\alpha^+ \cap K_\beta^-) \ge \dim_H K_\alpha^+ + \dim_H K_\beta^- - \dim_H \Lambda.$$

Moreover, it follows from Theorem 1.4.4 and Lemma 12.2.5 that

$$\dim_H(K_\alpha^+ \cap K_\beta^-) \le \dim_H K_\alpha^+ + \dim_H K_\beta^- - \dim_H \Lambda,$$

and thus,

$$\mathcal{D}(\alpha, \beta) = \dim_H K_\alpha^+ + \dim_H K_\beta^- - \dim_H \Lambda.$$

Using the identities (12.23) and (12.30), we obtain

$$\mathcal{D}(\alpha, \beta) = \dim_H (K_\alpha^+ \cap V^u(x)) + \dim_H (K_\alpha^- \cap V^s(x)).$$

The remaining statements follow now readily from Theorem 12.1.4. □

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